8.03 Fall 2004
Problem Set 2 Solutions

Solution 2.1: Take home experiment # 1 – Influence of mass on the damping of a pendulum

I (Igor Sylvester) made my pendulum about 65 cm long. I measured the time of 20 complete oscillations of the pendulum with the heavy weight ($20T_{\text{heavy}} = 33 \pm 1$ s) and the light weight ($20T_{\text{light}} = 32 \pm 1$ s). I also measured $t_{1/2}$ five times for both weights and calculated the best estimate and spread. The result for the heavy weight is $t_{1/2}^{\text{heavy}} = 150 \pm 7$ s ($\sim 4.5\%$ uncertainty) and for the light weight is $t_{1/2}^{\text{light}} = 31 \pm 1$ sec ($\sim 3\%$ uncertainty).

1. What is the relation between τ $(= \gamma^{-1})$ and the time you measured, $t_{1/2}$, that it takes for the amplitude to decrease by a factor of 2? The pendulum is described by

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = 0,$$

where $x(t)$ is the displacement of the mass from equilibrium, $\gamma = b/m$ and $\omega_0^2 = g/L$. We know that, in the under-damped case, this equation has the solution

$$x(t) = Ce^{-\frac{\gamma t}{2}} \cos(\omega t - \alpha),$$

where C and α are the initial amplitude and phase of oscillations, respectively and $\omega = \sqrt{\omega_0^2 - (\gamma/2)^2}$. Then, the amplitude of the oscillations is $Ce^{-\frac{\gamma t}{2}}$. If $t_{1/2}$ is the time the oscillator takes to half its amplitude, then

$$\begin{aligned}
Ce^{-\frac{1}{2}\gamma t_{1/2}} &= \frac{C}{2} \\
e^{-\frac{1}{2}\gamma t_{1/2}} &= \frac{1}{2} \\
\frac{1}{2}\gamma t_{1/2} &= \ln 2 \\
\Rightarrow t_{1/2} &= \frac{2\ln 2}{\gamma} \\
&= 2\tau \ln 2
\end{aligned}$$

.

2. How is an infinitesimal fractional change in the period related to an infinitesimal fractional change in the length? We know $T = 2\pi\sqrt{l/g}$. Differentiating once gives

$$dT = \frac{2\pi}{\sqrt{g}} \frac{dl}{2\sqrt{l}}.$$

Dividing these two relations gives

$$\frac{dT}{T} = \frac{1}{2}\frac{dl}{l}.$$

3. How well, fractionally, could you determine the period T? How accurately, in cm, would you have to reposition the mass to set T to the same value within your experimental ability to determine T?

If we can measure time with a watch with an accuracy of 1 s then we can measure 20 periods of oscillation within $20T \pm 1$ s. Hence, we can fractionally determine $dT/T = 1/20$. Using the relation derived earlier, we would need to position the weights within $dl/l = 1/10$ (10% accuracy). Then, we would need to adjust the pendulum's length with an accuracy of about 7 cm. Remember that the length of my pendulum was about 66 cm.

4. What was the Q of the oscillator in each case?

$Q = \omega_0/\gamma$ and $\omega = 2\pi/T \approx 3.8$ s^{-1} (3% uncertainty). Furthermore, $\gamma_{\text{heavy}} = 0.0092$ s^{-1} ($\sim$ 4.5% uncertainty) and $\gamma_{\text{light}} = 0.045$ s^{-1} ($\sim$ 3% uncertainty). Assuming $\omega \approx \omega_0$, $Q_{\text{heavy}} \approx 410$ ($\sim$ 5.5% uncertainty) and $Q_{\text{light}} \approx 85$ ($\sim$ 4% uncertainty).

Note: if the fractional uncertainties in a and b are $\delta a/a$ and $\delta b/b$ respectively, then the fractional uncertainty in a/b is $\sqrt{(\delta a/a)^2 + (\delta b/b)^2}$. Thus, a 3% uncertainty in a and 5% uncertainty in b result in a $\sim 5.8\%$ uncertainty in a/b.

5. Damping causes the actual frequency ω to fall below the undamped frequency ω_0. Should you be able to measure this frequency (or period) shift when you change to the lighter mass?

   The damped frequency is

   $$\omega = \sqrt{\omega_0^2 - (\gamma/2)^2},$$

   where $\omega_0^2 = g/l$ and $\gamma = b/m$. Changing to the lighter mass decreases m, which in turn increases γ thus it decreases ω . However, in this experiment the values for γ are so small that the difference between ω_0 and ω is negligible. For example, consider the following definition

   $$\omega = \omega_0\sqrt{1 - \frac{1}{4Q^2}}.$$

   The two frequencies are related by the $\sqrt{1 - 1/4Q^2}$ term. This factor is 0.9999993 for the heavy weight. Similarly, the factor is 0.999983 for the light weight. Hence, for this experiment, $\omega_0 \approx \omega$ to a very high degree of accuracy.

6. Were your damping times proportional to the mass? If they were not, do you think the discrepancy is due to the measurement accuracy or to other sources of damping?

   The weight of the heavy object, according to the instructions is 10 oz. This translates into a mass of 281 g. We did weigh the object and found a mass of 270 ± 1 g. The mass of the shell is 3 ± 1 g. That adds up to a total of 273 ± 2 g (0.7% uncertainty). The weight of the light object, according to the instruction is 1.5 oz. This translates into a mass of 42.2 g. We did weigh the object and found 42 ± 1 g. If we add the shell, we get 45 ± 2 g (4.5% uncertainty). The mass ratio is therefore $(273 \pm 2)/(45 \pm 2) = 6.07 \pm 0.27$ (4.5% uncertainty).

   The damping time ratio for the two masses is $t_{\text{heavy}}/t_{\text{light}} = (150 \pm 7)/(31 \pm 1) = 4.84 \pm 0.27$. Clearly this is substantially smaller than the mass ratio. Why?

   Notice that the heavy object sticks out above the spherical shell. That means that the damping coefficient, b, for the heavy weight is larger than for the light weight. If we approximate the portion of the heavy weight that sticks out above the shell by a sphere, we can use the results for spherical objects. Professor Lewin discusses the drag (v and v^2 terms) in detail in his lecture #12 (Fall 1999 – OCW) http://ocw.mit.edu/OcwWeb/Physics/8-01Physics-IFall1999/VideoLectures/index.htm

   For spheres, the damping coefficient (the one which is linear in v) is linearly proportional with the radius r. We made a guestimate of the radius of the approximated sphere that sticks out above the shell, and concluded that it is roughly 1/3 of that of the shell. Thus the corrected damping coefficient (shell plus the part that sticks out) is about 1.3 times that of the shell alone.

   If we take this into account, we find that the ratio of the damping times (which is the ratio of the reciprocals of γ for the two masses) becomes $(m_{\text{heavy}}/b_{\text{light}})/(m_{\text{light}}/b_{\text{heavy}}) = m_{\text{heavy}}/1.3m_{\text{light}} \approx 4.7$. This clearly explains why the damping times were not linearly proportional to the masses.

Solution 2.2: Driven oscillator with damping

Part (a)

An object of mass m is hung from a spring with spring constant 80 N/m. The resistive damping force on the object is given by $-bv$, where v is the velocity and $b = 4$ N m^{-1} sec. So the constants for the damped motion are

$$\gamma = \frac{b}{m} = 20\ s^{-1} \qquad \omega_0 = \sqrt{\frac{k}{m}} = 20\ s^{-1}$$

Let the oscillations of the spring be along the x axis. The spring force and damping force acting on the mass are

$$F_{restoring} = -kx \qquad F_{damping} = -bv = -b\dot{x}$$

Newton's 2nd law:

$$\begin{aligned} m\ddot{x} &= F_{net} = F_{restoring} + F_{damping} = -kx - b\dot{x} \\ \ddot{x} &= -\frac{k}{m}x - \frac{b}{m}\dot{x} \end{aligned}$$

Hence the differential equation describing the motion of the mass is:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

or

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0 \tag{1}$$

The frequency and period of such damped oscillations are:

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = 300 \qquad \Rightarrow \qquad \omega = 10\sqrt{3}\ s^{-1} = 17.3\ s^{-1}$$

$$T_{period} = \frac{2\pi}{\omega} = \frac{2\pi}{10\sqrt{3}}\ s \approx 0.36\ s$$

Part (b)

The equation of motion for this harmonically driven damped oscillator is:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\sin(\omega t) \tag{2}$$

The amplitude of oscillations in the steady state is given by the formula :

$$\begin{aligned} A(\omega) &= \frac{F_0/m}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}} \\ &= \frac{2/0.2}{[(20^2 - 30^2)^2 + (20\cdot 30)^2]^{\frac{1}{2}}} \\ &= 0.0128\ m = 1.28\ cm \end{aligned} \tag{3}$$

substituting values for ω_o,F_0,ω and γ.

Part (c)

In Fig. 1, A is the equilibrium level of the top end of the spring and B is the equilibrium level of the mass. $X = X_0\cos(\omega t)$ is the harmonic displacement of the top end of the spring from its equilibrium position A, and x is the displacement of the spring from its equilibrium position.

The spring force and the damping force acting on the mass are given by

$$F_{restoring} = +k(X - x) \qquad F_{damping} = -bv = -b\dot{x}$$

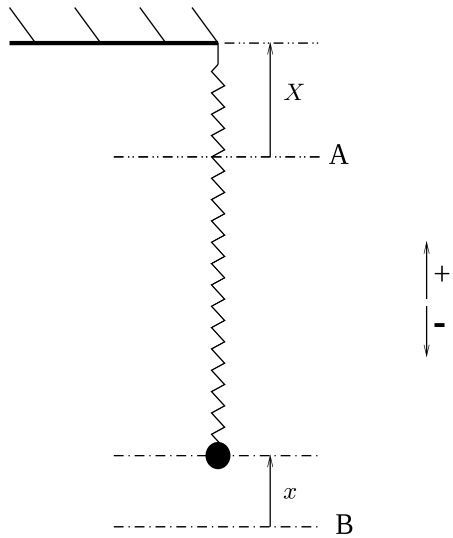

FIG. 1: Plot of spring with harmonic oscillations applied to the top end of the spring

For $X > x$ the spring force is in the + direction. Newton's 2nd law:

$$\begin{aligned} m\ddot{x} &= F_{net} = F_{restoring} + F_{damping} = +k(X - x) - b\dot{x} \\ \ddot{x} &= \frac{k}{m}(X - x) - \frac{b}{m}\dot{x} \\ \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x &= \frac{k}{m}X_0\cos(\omega t) \end{aligned} \tag{4}$$

This equation has the same form as Eq. 2. The harmonic displacement of the top end of the spring is equivalent to the application of a driving force with amplitude $F_0 = kX_0$.

Part (d)

The amplitude of the mass in steady state is:

$$A(\omega) = \frac{kX_0/m}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}} \tag{5}$$

substituting values $\omega_o = 20\ \mathrm{s}^{-1}$, $k = 80$ N/m, $\omega = 0, 30, 300\ \mathrm{s}^{-1}$ and $\gamma = 20\ \mathrm{s}^{-1}$.

$$A(\omega) = \frac{0.4/0.2}{[(20^2 - \omega^2)^2 + (20\omega)^2]^{\frac{1}{2}}}$$

$$A(0) = 0.5\ cm \quad A(30) = 0.256\ cm \quad A(300) = 0.00223\ cm \tag{6}$$

Solution 2.3: (French 4-6) Seismograph

Part (a)

The displacement of mass M relative to the earth is y and η is the displacement of the earth's surface relative to the distant stars. Let x be the distance of mass M relative to the distant stars.

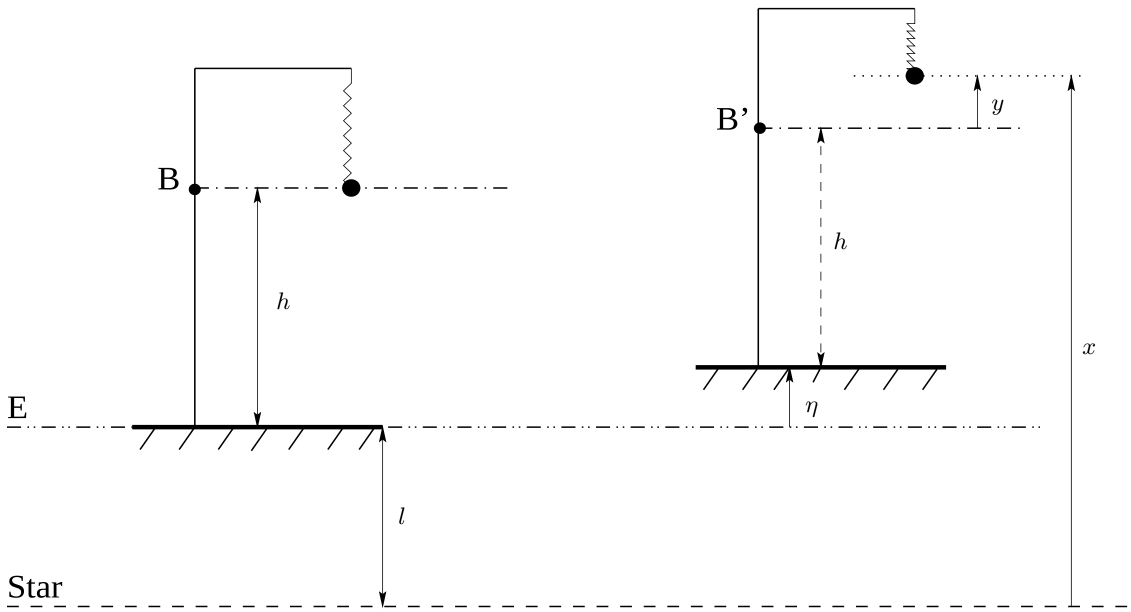

FIG. 2: Plot of seismometer before and during the earthquake

Left Figure: The horizontal dashed line through E is the equilibrium position of the earth relative to the star. The horizontal dashed line through B is the equilibrium position of the mass relative to the star. It is also the equilibrium position of the mass relative to the seismometer.

Right Figure: The dashed line through E, is the same as in the left figure. B is now a distance η farther away from the star than B (we indicate this with B'). The dashed line through B' is no longer the equilibrium position of the mass relative to the star, but it is the equilibrium position relative to the seismometer.

We can see from the figures that:

$$\begin{aligned} x &= l + y + h + \eta \\ \ddot{x} &= \ddot{\eta} + \ddot{y} \end{aligned}$$

Newton's 2nd law only applies to an inertial reference frame. The acceleration of M is $\ddot{x}$. However, the spring force and the damping force depend on the displacement and velocity relative to the Earth (i.e. relative to B'). The amount by which the length of the spring changes is y in both reference frames (that of the star and that of the seismograph). Thus the magnitude of the spring force is ky. Since it is assumed that the air inside the closed box of the seismograph follows the motion of the Earth, the damping force is $-b\dot{y}$. Notice, if the air does not follow the Earth then the damping force would be $-b(\dot{y} + \dot{\eta})$. Hence:

$$\begin{aligned} M\ddot{x} &= -ky - b\dot{y} \\ 0 &= \ddot{\eta} + \ddot{y} + \frac{k}{M}y + \frac{b}{M}\dot{y} \end{aligned}$$

$$-\ddot{\eta} = \ddot{y} + \gamma\dot{y} + \omega_0^2 \qquad or \qquad \frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = -\frac{d^2\eta}{dt^2}$$

$$where \qquad \gamma = \frac{b}{m} \qquad \omega_0^2 = \frac{k}{m} \tag{7}$$

QED

Part (b)

Steady state solution for y when $\eta = Ccos\omega t$.

$$\begin{aligned}
\eta &= Ccos(\omega t) \\
\frac{d^2\eta}{dt^2} &= -C\omega^2 cos(\omega t) \\
\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y &= C\omega^2 cos(\omega t) \qquad (8)
\end{aligned}$$

To solve the equation using the complex exponential method we reframe the above equation as follows

$$\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z = C\omega^2 e^{i\omega t}$$

Let $z = Ae^{i(\omega t-\delta)}$ be the solution to the above equation. Now $y = Re(z)$. Substituting these in Eq. 8.

$$\begin{aligned}
(-\omega^2 A + i\gamma\omega A + \omega_0^2 A)e^{i(\omega t-\delta)} &= C\omega^2 e^{i\omega t} \\
(\omega_0^2 - \omega^2)A + i\gamma\omega A &= C\omega^2 e^{i\delta}
\end{aligned}$$

Equating the real and imaginary parts of the equation we get:

$$\begin{aligned}
(\omega_0^2 - \omega^2)A &= C\omega^2 cos\delta \\
\gamma\omega A &= C\omega^2 sin\delta
\end{aligned}$$

Therefore the steady state solution for y is as follows

$$y = Acos(\omega t - \delta) \qquad (9)$$

where

$$\begin{aligned}
A(\omega) &= \frac{C\omega^2}{[(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}} \qquad (10) \\
tan\ \delta(\omega) &= \frac{\gamma\omega}{\omega_0^2 - \omega^2} \qquad (11)
\end{aligned}$$

Behavior of $A(\omega)$ for various values of ω

$$\begin{aligned}
\omega \to 0 &\qquad A \to 0 \\
\omega \to \omega_0 &\qquad A \to QC \\
\omega \to \infty &\qquad A \to C
\end{aligned}$$

Part (c)

The graph of the amplitude A of the displacement y as a function ω is shown in Fig. 3.

Part (d)

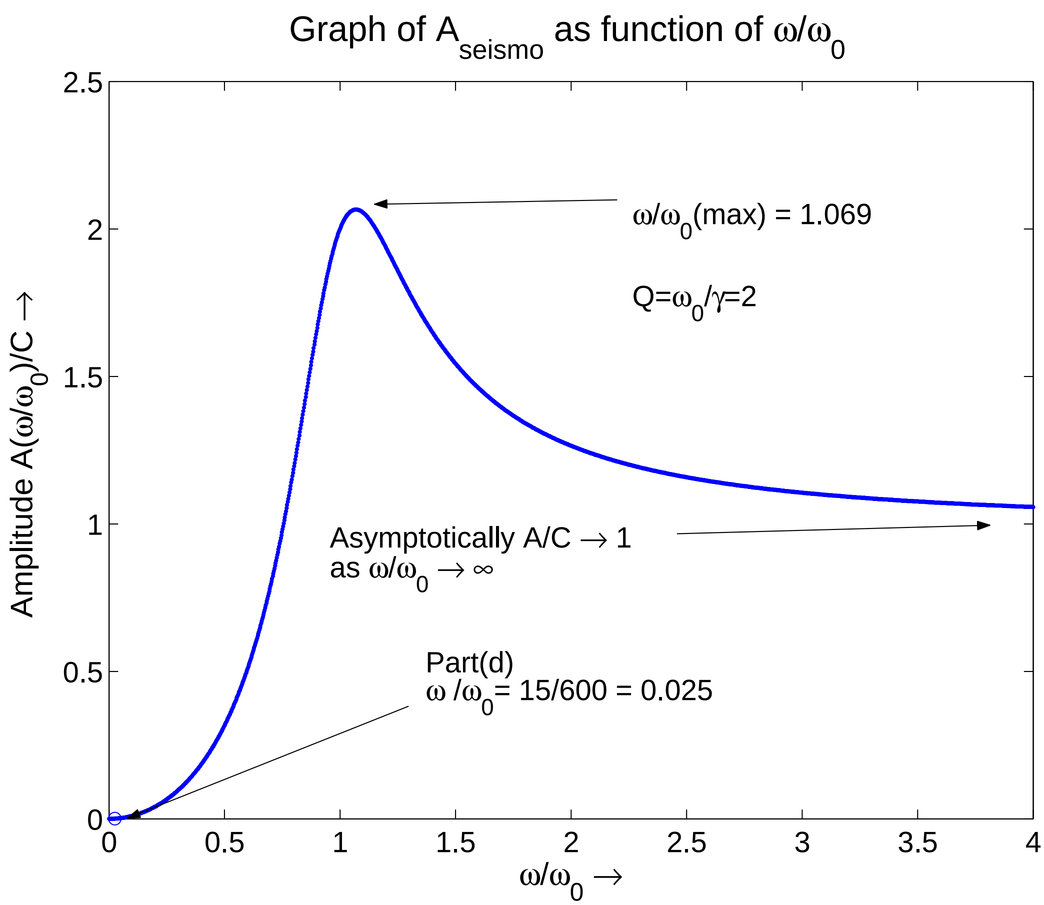

FIG. 3: Graph of Amplitude $A(\omega)$ in units of C versus ω/ω_0 [Note: Q=ω_0/γ is taken to be 2]

Period of the Seismograph T_s is $30s$ and Q is 2.

$$\begin{aligned} T_s &= 2\pi/\omega_0 = 30s \\ \omega_0 &= \frac{2\pi}{30} = \frac{\pi}{15}\ rad/s \\ \gamma &= \frac{\omega_0}{Q} = \frac{\pi/15}{2} = \frac{\pi}{30}\ rad/s \end{aligned}$$

Now the time period of oscillations of the earth's surface is 20 min and the amplitude of maximum acceleration is $10^{-9} m/s^2$.

$$\begin{aligned} \omega &= \frac{2\pi}{T_s} = \frac{2\pi}{1200} = \frac{\pi}{600}\ rad/s \\ a_{max} &= C\omega^2 = 10^{-9}\ m/s^2 \\ C &= \frac{a_{max}}{\omega^2} = 3.6 \times 10^{-5}\ m \end{aligned}$$

Substituting values for ω, ω_0, γ and C in the equation for amplitude A we get:

$$\begin{aligned} A(\omega) &= \frac{C\omega^2}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}} \\ A &= 2.28 \times 10^{-8}\ m \end{aligned}$$

Notice that C (amplitude of the Earth's oscillations) is about 1600 times larger than A. It seems to us that this is a very poorly designed seismometer. Values of A of the order of 2.3×10^{-8} m must be observable for this tremor to be detected. If the frequency of the oscillations $\omega \gg \omega_0$ the value of $A \to C$ (see Fig. 3). The amplitude of the earthquake oscillations can then directly be read off the seismometer.

Part (e)

Problem 2.3 and 2.2 are very different. In Fig. 4, we show the amplitude $A(\omega)$ versus ω in Problem 2.2. Compare this to the plot in Problem 2.3 as shown in Fig. 3

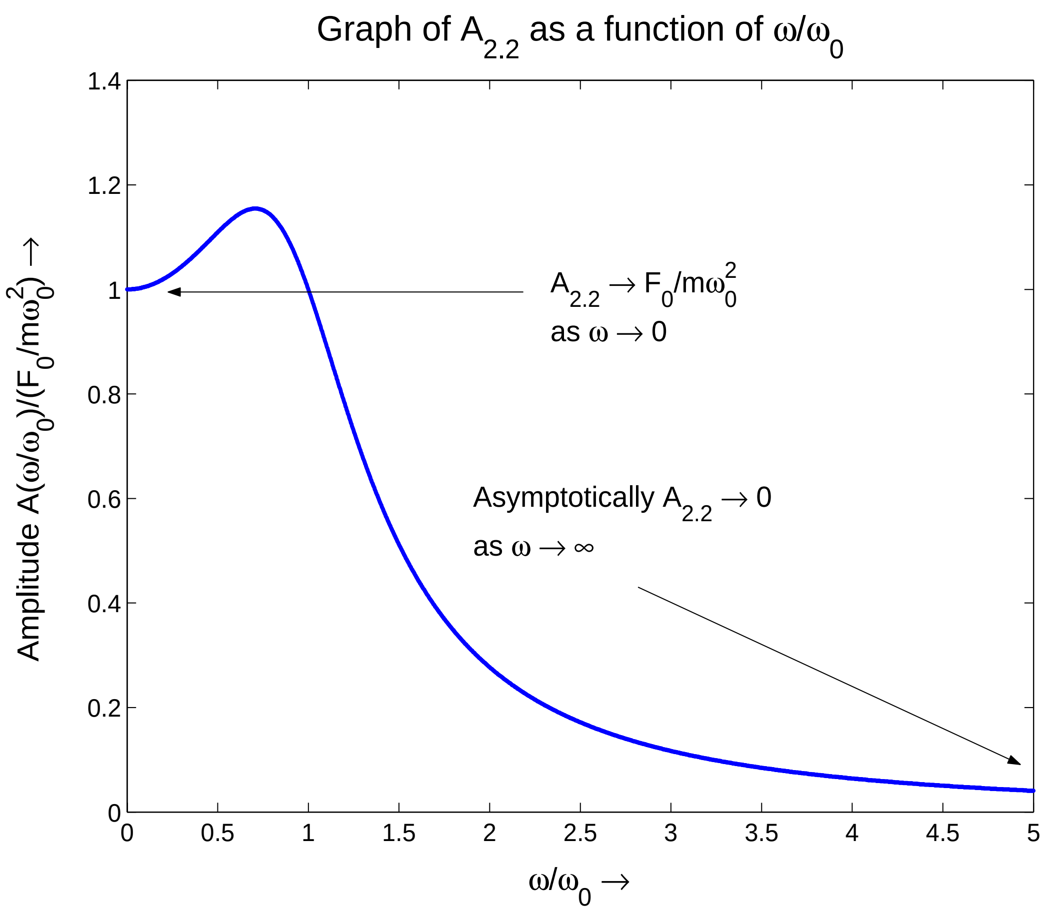

FIG. 4: Plot of A(ω) vs. ω from Problem 2.2 [Note: $\omega_0 = 20$ Hz,$\gamma = 20$ Hz, and $F_0/m = 2$ N/kg]

This difference is best demonstrated by comparing their amplitudes at very low (near zero) and very high frequencies. Let the amplitude in Problem 2.2 be $A_{2.2}$ and the amplitude of the seismometer be A_{seismo}. Now

$$\omega \to 0 \qquad \Rightarrow \qquad A_{2.2} \to \frac{F_0}{m\omega_0^2} \qquad A_{seismo} \to 0 \tag{12}$$

$$\omega \to \infty \qquad \Rightarrow \qquad A_{2.2} \to 0 \qquad A_{seismo} \to C \tag{13}$$

$$\omega \to \omega_0 \qquad \Rightarrow \qquad A_{2.2} \to \frac{F_0}{\gamma m\omega_0} \qquad A_{seismo} \to \; QC \tag{14}$$

As you can see, there is a major difference between harmonically displacing the top end of the spring and harmonic oscillations of the earth.

Solution 2.4: (French 4-10) Power dissipation

Part (a)

Let dW be the work done against the damping force in time dt. Now the work done is the dot product of the force and the distance over which it is applied, $dW = F_{anti-damping}dx = bv \cdot dx$. Hence the instantaneous rate of doing work against the damping force is:

$$P = \frac{Work\ Done}{Time\ Taken} = \frac{dW}{dt} = bv\frac{dx}{dt} = bv^2 \tag{15}$$

QED

Part (b)

The equation of motion is of the form $x = A\cos(\omega t - \delta)$, hence the mean power dissipated can be calculated from part (a) as shown below:

$$\begin{aligned}
\bar{P} &= (b\dot{x}^2)|_T = [b(-A\omega\sin(\omega t - \delta))^2]|_T \\
&= bA^2\omega^2[\sin^2(\omega t - \delta)]|_T = bA^2\omega^2\frac{1}{2} \qquad (\ Since\ \sin^2(\omega t - \delta)|_T = \frac{1}{2}\ ) \\
&= \frac{bA^2\omega^2}{2}
\end{aligned}$$

Part (c)

The value of A for any arbitrary frequency is given by the expression shown below

$$A(\omega) = \frac{F_0/m}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}} \qquad \Rightarrow \qquad A(\omega)^2 = \frac{F_0^2/m^2}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

$$\begin{aligned}
\bar{P}(\omega) &= \frac{b\omega^2}{2}\frac{F_0^2/m^2}{\omega^2\omega_0^2[(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + (\frac{\gamma}{\omega_0})^2]} \\
&= \frac{bF_0^2}{2\omega_0^2m^2}\frac{1}{(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + (\frac{\gamma}{\omega_0})^2} \\
&= \frac{F_0^2\gamma}{2k}\frac{1}{(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + (\frac{\gamma}{\omega_0})^2} \\
\bar{P}(\omega) &= \frac{F_0^2\omega_0}{2kQ}\frac{1}{(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0})^2 + \frac{1}{Q^2}}
\end{aligned} \tag{16}$$

Hence the mean rate of power dissipation is shown to be same as in Eq. French 4-23.

Solution 2.5: Transient behavior

Part (a)

The period of free oscillations can be measured off the graph to be T_0 approximately 4 sec, hence

$$\nu_0 = \frac{1}{T_0} \approx \frac{1}{4} \approx 0.25\ Hz \qquad \omega_0 \approx 0.5\pi\ rad/s \tag{17}$$

Part (b)

The homogenous solution is given by $x(t) = x(t=0)e^{-\gamma t/2}\cos(\omega t + \phi)$. To determine we use the envelope of the exponential decay. A couple of points on the exponential decay envelop are measured to be $x(t \approx 0.7) \approx -10.5$ m and $x(t \approx 3.0s) \approx +6.2$ m. Now

$$\begin{aligned}\left|\frac{x(t=3.0\ s)}{x(t=0.7\ s)}\right| &= e^{-2.3\gamma/2} = \frac{6.2}{10.5} \\ \Rightarrow \quad \gamma &= \frac{2}{2.3}\ln\left(\frac{10.5}{6.2}\right) \approx 0.46\ rad/s\end{aligned}$$

Hence the damping coefficient $b = m\gamma \approx 0.46$ Ns/m. As pointed out by Professor Ketterle, a much better way would be to subtract the steady state solution from the total curve and then to derive γ from the decay of the remaining curve. That would certainly give a more accurate value. However, we did not do that - sorry. This may explain why our value for γ is more than 20% off the value that was used to generate the curve; see part (e) and Figure 5.

Part (c)

The frequency of the driving force can be measured quite accurately with the period of the steady state solution. The period appears to be 5 cycles in last 5 seconds or $\nu \approx 1$ Hz . Hence

$$T = \frac{1}{\nu} \approx 1\ s \qquad \omega = 2\pi\nu \approx 2\pi\ rad/s \tag{18}$$

The frequency of the driving force is ~ 1 Hz. It is four times larger than the frequency of the free oscillations.

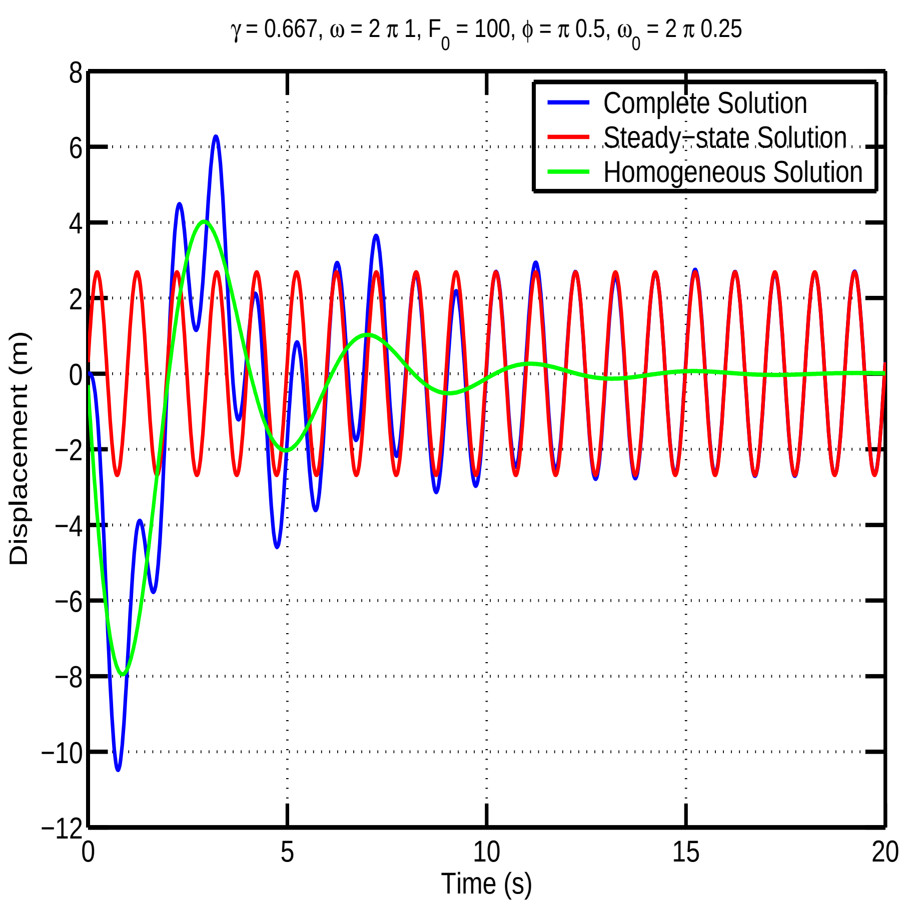

FIG. 5: Plot of position of the mass m as a function of time t showing the various components of its motion

Part (d) The amplitude of the steady state response of the oscillator can be measured quite accurately

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} = 2.5\ m$$

from which we can calculate the amplitude of the driving force

$$\begin{aligned} F_0 &= mA(\omega)\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} \\ &= 1 \times 2.5\sqrt{16\pi^4(0.25^2 - 1^2)^2 + 4\pi^2(0.46^2 \times 1^2)} \\ &\approx 93\ N \\ &\simeq 90\ N \end{aligned} \tag{19}$$

Part (e)

We find the initial phase of the driving force by extending the steady-state part back to $t = 0$. The zero crossings in the steady state solution are at $t = 10, 15$ and 20 sec. Thus it also crosses zero at t=0 sec, and the steady state displacement starts off in the positive direction. We also know that when $\omega \gg \omega_0$, the displacement $x(t)$ is π radians out of phase with the driving force $F(t) = F_0 \cos(\omega t + \phi)$. Thus at $t = 0$ the driving force must be zero and must increase in the negative direction. Thus

$$F(t = 0) = F_0 \cos\phi = 0 \qquad\qquad F(t = +\epsilon) < 0$$

$$\Rightarrow \quad \phi \approx \frac{\pi}{2}\ radians \tag{20}$$

Fig. 5 shows a graph of the transient (green), the steady state (red) and the composite (blue). At the top of the plot we list the input parameters that we used in preparing this problem. Compare them with the approximate values that we derived from the blue plot. We were dead on in the case of ω, ω_0 and ϕ. We were within 10% of F_0, but our estimate of γ was off by more than 20%.

Solution 2.6: Driven RLC circuit

Part (a)

Potential differences across the resistor and the capacitor are as follows:

$$V_R = IR = R\frac{dq}{dt} \qquad V_C = \frac{q}{C}$$

Faraday's Law states

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\phi_B}{dt}$$

The inductor has no ohmic resistance. Thus the $\int \vec{E} \cdot \vec{dl}$ in going through the wire of the inductor from one end to the other end is zero. The closed loop integral going into the direction of the current then becomes

$$IR + V_C - V_0\cos(\omega t) = -L\frac{dI}{dt}$$

Notice: Kirchhoff's "volage" rule does *NOT* hold as the E field here is non-conservative. Substituting $I = dq/dt$ and $dI/dt = d^2q/dt^2$, the differential equation for charge on the capacitor is

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{L}\frac{q}{C} = \frac{V_0}{L}\cos(\omega t) \quad \Rightarrow \quad \frac{d^2q}{dt^2} + \gamma\frac{dq}{dt} + \omega_0^2 q = \frac{V_0}{L}\cos(\omega t) \tag{21}$$

$$\frac{R}{L} = \gamma \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

We differentiate the above equation to find the equation for current I

$$\frac{d^2I}{dt^2} + \gamma\frac{dI}{dt} + \omega_0^2 I = -\frac{V_0}{L}\omega\sin(\omega t) \tag{22}$$

Part (b)

To solve for $q(\omega, t)$, we use the fact that Eq. 21 has the same form as Eq. 2 in Problem 2.2(b). Hence the solution is:

$$q(\omega, t) = q_0(\omega)\cos(\omega t - \delta) \tag{23}$$

$$q_0(\omega) = \frac{V_0/L}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}}$$

$$\tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

Part (c)

We calculate $I(\omega, t)$ by differentiating our results from above for $q(\omega, t)$:

$$I(\omega, t) = -\omega q_0(\omega)\sin(\omega t - \delta) = -I_0(\omega)\sin(\omega t - \delta) \tag{24}$$

$$|I_0(\omega)| = \frac{\omega V_0/L}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}}$$

$$\tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

The equation for $I_0(\omega)$ is often written in the form

$$I_0 = \frac{V_0}{\sqrt{R^2 + (X_L - X_c)^2}} \tag{25}$$

where $X_C = 1/\omega C$ and $X_L = \omega L$ are the capacitive and inductive reactances respectively.

At resonance, $delta = \pi/2$. That means that the driving voltage is IN PHASE with the current. Because $cos\omega t = -sin(\omega t - \pi/2)$. As menioned in lectures, at resonance, the circuit behaves as if there is no C and no L. Thus Ohm's Law is at work which dictates that the voltage and the current are in phase. Consequently, for low values of ω when $\delta = 0$, the current is leading the voltage by a phase angle $\pi/2$ which corresponds to a quarter of a period (the capacitor rules!), and for very high ω, the current is lagging the driving voltage by a quarter of a period (the self-inductor rules!).

Part (d)

Substituting values for V_0 , R, L , and C; the plot for current I as a function of ω is shown in Fig. 6.

Part (e)

We can see from Eq. 25 that for the current I_0 through the circuit to be maximum, $Z = \sqrt{R^2 + (X_L - X_c)^2}$ has to be minimized. Thus

$$\begin{aligned} X_L = X_C \qquad &\Rightarrow \qquad \omega L = \frac{1}{\omega C} \\ \omega_{Imax} &= \frac{1}{\sqrt{LC}} = \omega_0 = 3.162 \times 10^4 \ rad/s \end{aligned} \tag{26}$$

At frequency $\omega_{Imax} = 3.162 \times 10^4 \ rad/s$, the current through the circuit is maximum.
The quality factor Q for the system is

$$\gamma = \frac{R}{L} = 500 \ rad/s \qquad Q = \frac{\omega_0}{\gamma} = \frac{L}{R}\frac{1}{\sqrt{LC}} = 63.2 \tag{27}$$

The high value of $Q = 63.2$ explains the sharp peak around $\omega = 3.162 \times 10^4$ rad/s.

Part (f)

The plot for charge q as a function of ω is shown in Fig. 7.

Part (g)

To find the ω at which q_0 is maximum, we differentiate its value from Eq. 23 with respect to ω and equate it to 0

$$\begin{aligned} \frac{d[q_0(\omega)]}{d\omega} = 0 &= \frac{d}{d\omega}\left[\frac{V_0/L}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}}\right] \\ 0 &= -\frac{V_0}{2L}\frac{4\omega(\omega^2 - \omega_0^2) + 2\omega\gamma^2}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{3}{2}}} \\ \Rightarrow \qquad 4\omega(\omega^2 - \omega_0^2) &= -2\omega\gamma^2 \\ \omega_{qmax} &= \sqrt{\frac{1}{2}(2\omega_0^2 - \gamma^2)} = 3.160 \times 10^4 \ rad/s \end{aligned} \tag{28}$$

At frequency $\omega_{qmax} = 3.160 \times 10^4 \ rad/s$, the charge on the capacitor is maximum.

The frequency ω_{qmax} at which charge q_0 on the capacitor is a maximum, is only slightly lower than the frequency ω_{Imax} at which the current I_0 through the circuit is maximized. The difference is very small as Q is very high (~ 63).

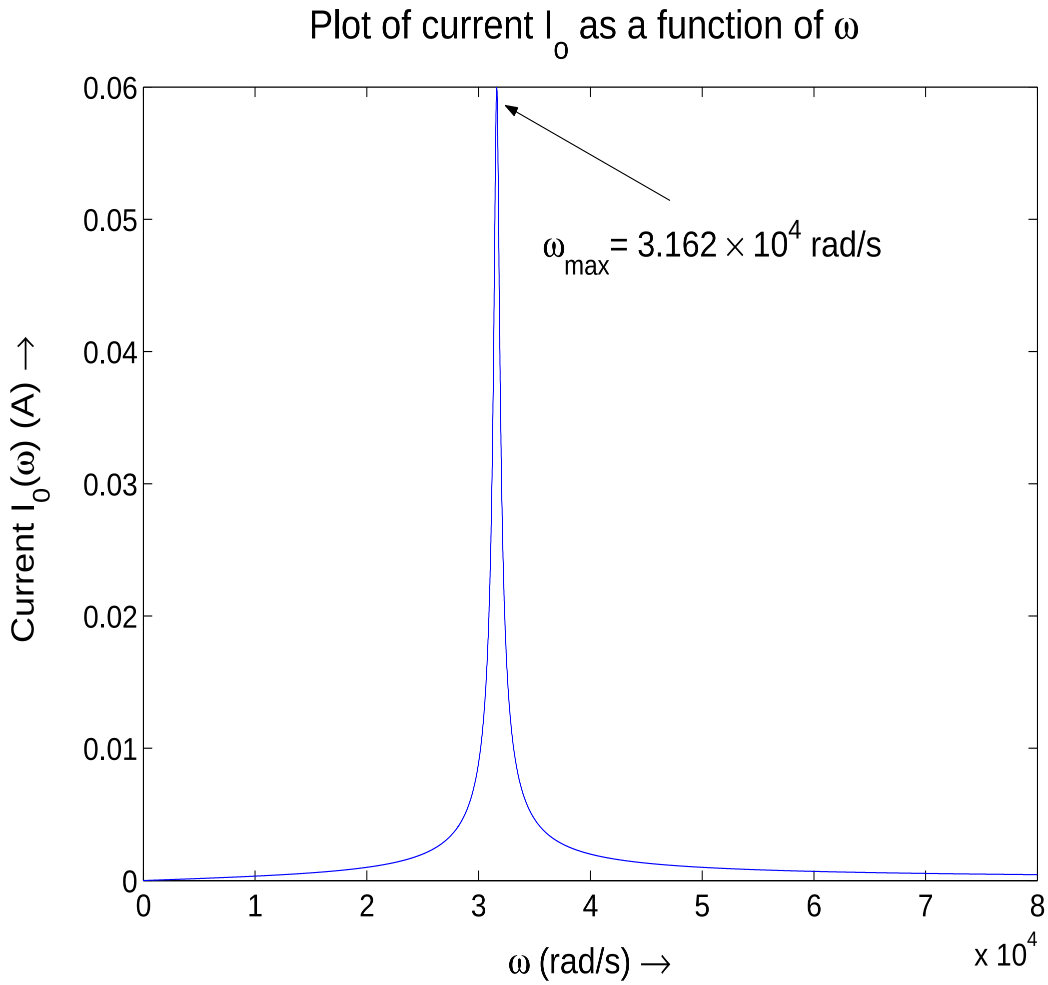

FIG. 6: Plot of current as a function of ω [$V_0 = 3$ V, $R = 50\ \Omega$, $L = 100$ mH, and $C = 0.01\ \mu$F]

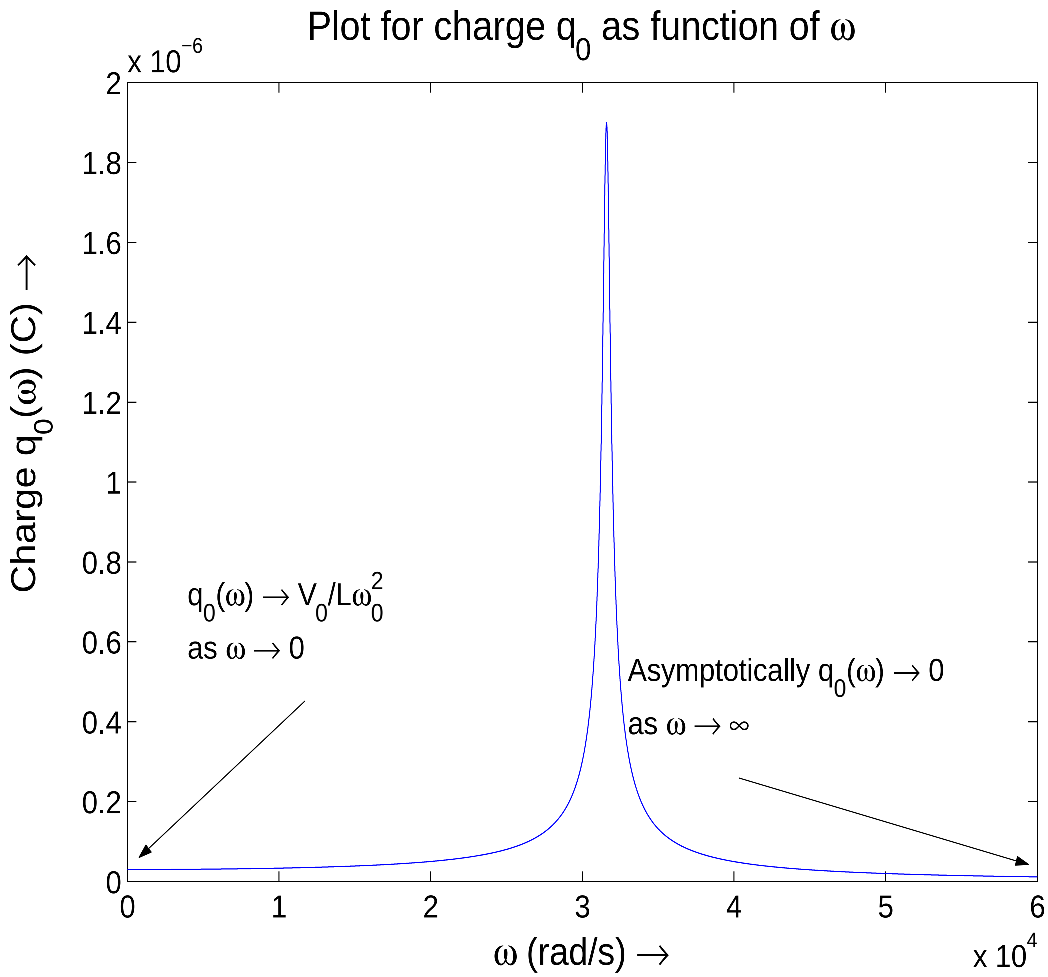

FIG. 7: Plot of charge as a function of ω [$V_0 = 3$ V, $R = 50\ \Omega$, $L = 100$ mH, and $C = 0.01\ \mu$F]