MIT 8.03 Fall 2004 – Solutions to Problem Set 5

Problem 5.1 — Piano galore

(a) The frequency of the n-th mode of a string is $\nu_n = \omega_n/2\pi = n\sqrt{T}/2L\sqrt{\mu}$. Differentiating with respect to T gives

$$
\begin{aligned}
\frac{d\nu_n}{dT} &= \frac{n}{4L}\sqrt{\frac{1}{T\mu}} \\
&= \left(\frac{n}{2L}\sqrt{\frac{T}{\mu}}\right)\frac{1}{2T} \\
&= \frac{1}{2T}\nu_n
\end{aligned}
$$

We know that $n = 1$, $T = 250$ N, $\nu_{C_5} = 512$ Hz and $d\nu = 0.5$ Hz. Thus, $dT = (0.5)(500)/512$ N $\approx$ 0.5 N.

(b) Pianos have 88 keys. Many notes have two strings and many have three; some have only one string. A Steinway grand piano has a total of 216 strings. This translates into $F = 216 \times 250\ N \approx 5.4\,10^4\ N$. This is huge; it's about the weight of a mass of 54 thousand kg (54 tons)!!

(c) The G_5 will excite the second harmonic of C_4 and you will hear G_5. The fundamental of G_5 will not excite G_6. However, the second harmonic of G_5 will excite G_6 and you will hear G_6.

(d) A note which is a higher harmonic of G_5 will be excited (eg. G_6, D_7, G_7, B_7). Also a note below G_5 which has G_5 as one of its higher harmonics will be excited (e.g. $G_4, C_4, G_3, E_3^b, C_3$, etc.).

(e) No string is perfectly flexible and perfectly continuous. Furthermore, the restoring force on the string is linear only to a first approximation, so it is not possible for the strings to possess harmonics in perfect multiples of each other. Very shortly we will learn that the velocity is a function of frequency (or λ); a phenomenon called dispersion. So far we always assumed ideal strings for which $v = \sqrt{T/\mu}$ (independent of ν).

There is another reason for the difference in tone between G_5 and the 6^{th} harmonic of C_3: a piano which is "in tune" is not tuned according to our scientific scale. The octaves are tuned in perfect multiples of 2 (frequency) but all other intervals are slightly altered. The perfect fifth is not so perfect after all. For more information see Waves (Berkeley Physics Course Vol. 3), by Crawford, problem 2.6 pp 91-93.

(f) They had better go away since the beats are the result of a superposition of sinusoidals of the two nodes.

Problem 5.2 – Holes in woodwind instruments

(a) With holes C and B closed, the pipe is 37 cm long, open at both ends. Therefore,

$$
\begin{aligned}
\lambda &= 2L = 74\,\text{cm} \\
\Rightarrow \nu &= \frac{v}{\lambda} = 446\,\text{Hz}.
\end{aligned}
$$

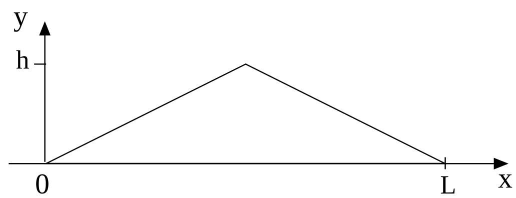
(b) If the holes are large enough this is a pipe of length 18.5 cm, open at both ends. Thus, $\nu = 892$ Hz.

(c) With only hole B closed, the effective length of the pipe is AC so $\lambda = 2(27.7\text{ cm}) = 55.4$ cm. Hence, $\nu = 600$ Hz.

(d) With neither B or C closed, L is now approximately 18 cm, thus $\lambda = 2(18.5 \text{ cm}) = 37$ cm. Hence, $\nu = 892$ Hz.

Problem 5.3 (French 6-12) — Plucked string

A sketch of the string is shown below.



(a) Remember that the kinetic energy density of a wave $y(x,t)$ in a string is

$$\frac{dK}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2,$$

and the potential energy density is

$$\frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2.$$

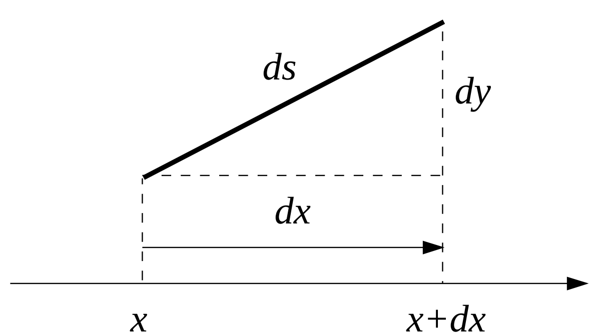
Here μ is the mass density and T is the tension in the string.

Then, the total energy of the string at $t = 0$ is

$$\begin{aligned}
E &= K + U = U \quad (K = 0 \text{ at } t = 0) \\
&= \int \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx \\
&= \frac{1}{2}T\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx \\
&= \frac{1}{2}TL\left(\frac{2h}{L}\right)^2 \\
E &= \frac{2h^2T}{L}.
\end{aligned}$$

Since energy is conserved (we ignore any form of damping) the energy at $t = 0$ is the same as the energy at later times.

Alternatively, we can calculate the potential energy of the string directly. The potential energy can be calculated by finding the amount by which the string, when deformed, is longer that when it is straight. This extension, multiplied by the assumed constant tension T, is the work done by us in the deformation. A displaced infinitesimal segment of a string is shown in the figure below.



Thus, for the segment, we have

$$dU = T(ds - dx),$$

where

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= dx\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \end{aligned}$$

If we assume that the transverse displacements are *small*, so that $\partial y/\partial x \ll 1$, we can approximate the above expression using the binomial expansion to two terms:

$$ds - dx \approx \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx.$$

Therefore,

$$\begin{aligned} dU &\approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx \\ \Rightarrow \frac{dU}{dx} &= \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2. \end{aligned}$$

(b) From our choice of coordinates, the shape of the string and that of all subsequent oscillations are odd functions. Hence, we can apply a Fourier transform to decompose the motion of the wave into sine functions only. They will have the form $y_n(x,t) = A_n \sin \omega_n t$, where A_n is the amplitude of the n-th harmonic, $\omega_n = n\omega_1$ and ω_1 is the angular frequency of the first harmonic (fundamental). The initial shape of the string repeats at an angular frequency of ω_1 because all harmonics repeat at an integer multiple of the first harmonic.

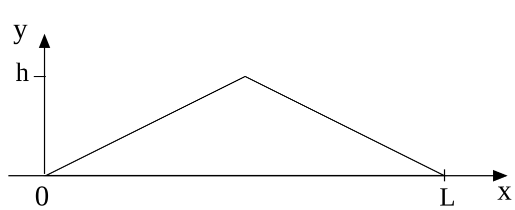
We can compute ω_1 from the relation $\omega_n = k_n v$. We know that the first harmonic has a wavelength $\lambda_1 = 2L$. Hence, $k_1 = 2\pi/\lambda_1 = \pi/L$. Therefore, $\omega_1 = \pi v/L = \pi/L\sqrt{T/\mu}$. Then the initial pulse shape repeats every $2L\sqrt{\mu/T}$ seconds. Notice that this is the travel time of a pulse from one end of the string to the other, and back.

Problem 5.4 — Fourier analysis

(a) The function is

$$y(x) = \begin{cases} \frac{2h}{L}x & \text{if } 0 \le x < L/2 \\ -\frac{2h}{L}x + 2h & \text{if } L/2 \le x \le L, \end{cases}$$

and a sketch of $y(x)$ is shown below



The most generic Fourier expansion is

$$y(x) = \sum_{n=0}^{\infty} A_n \cos(k_n x) + B_n \sin(k_n x)$$

Since $f(0) = 0$, all cosine terms will vanish. Furthermore,

$$\begin{aligned} y(L) &= 0 \\ \sum_{n=0}^{\infty} B_n \sin(k_n L) &= 0. \end{aligned}$$

Since, in general, $B_n \neq 0$ then,

$$\begin{aligned} \sin(k_n L) &= 0 \\ \Rightarrow k_n L &= n\pi \\ k_n &= \frac{n\pi}{L}. \end{aligned}$$

Hence, the Fourier expansion of $y(x)$ is

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right).$$

Notice that the sum starts at $n = 1$. The $n = 0$ term equals zero so it does not contribute. We can find the value of B_n by multiplying both sides by $\sin(m\pi x/L)$ and integrating with respect to x:

$$\begin{aligned} \int_0^L \sin\left(\frac{m\pi}{L} x\right) y(x)\, dx &= \int_0^L \sin\left(\frac{m\pi}{L} x\right) \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right)\, dx \\ \int_0^L \sin\left(\frac{m\pi}{L} x\right) y(x)\, dx &= \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right)\, dx. \end{aligned}$$

We recall the orthogonality property of the sine function,

$$\int_0^L \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n. \end{cases}$$

Hence,

$$\int_0^L \sin\left(\frac{m\pi}{L} x\right) y(x)\, dx = B_m \frac{L}{2}.$$

The Fourier coefficients then are

$$\begin{aligned} B_n &= \frac{2}{L} \int_0^L \sin\left(\frac{n\pi}{L} x\right) y(x)\, dx \\ &= \frac{2}{L} \left[\int_0^{L/2} \sin\left(\frac{n\pi}{L} x\right) \frac{2h}{L} x\, dx + \int_{L/2}^L \sin\left(\frac{n\pi}{L} x\right) \left(-\frac{2h}{L} x + 2h\right)\, dx \right] \\ &= \frac{4h}{n^2\pi^2} \left[2\sin\left(\frac{n\pi}{2}\right) - \underbrace{\sin(n\pi)}_{=0} \right] \\ &= \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

A few values of B_n are

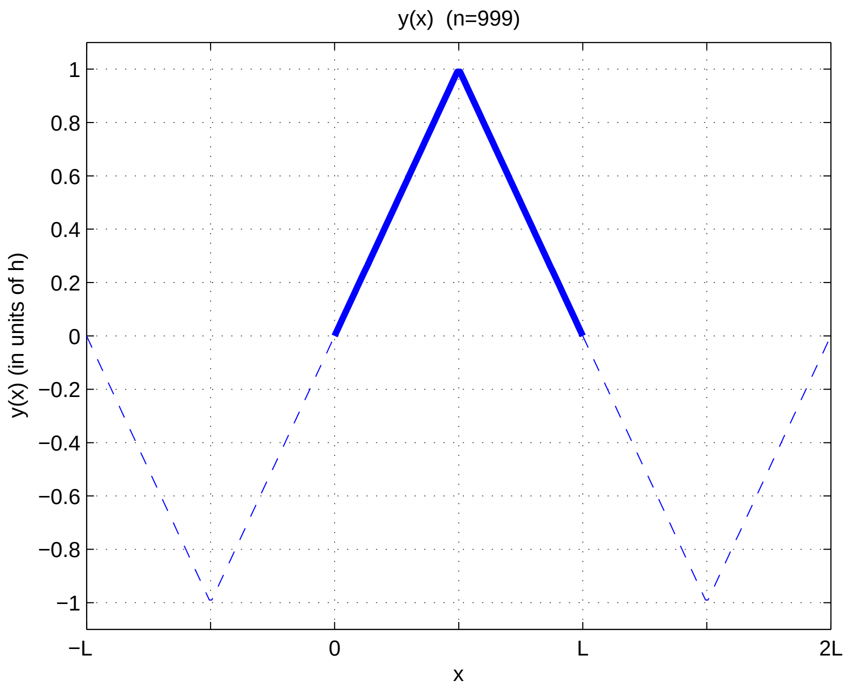
$$B_1 = \tfrac{8h}{\pi^2} \quad B_3 = -\tfrac{8h}{9\pi^2} \quad B_5 = \tfrac{8h}{25\pi^2}.$$

Note that B_n is zero for all even n and that the sign of B_n alternates for odd n. We could have predicted that. Why?

The Fourier expansion of $y(x)$ then is

$$y(x) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{L}x\right).$$

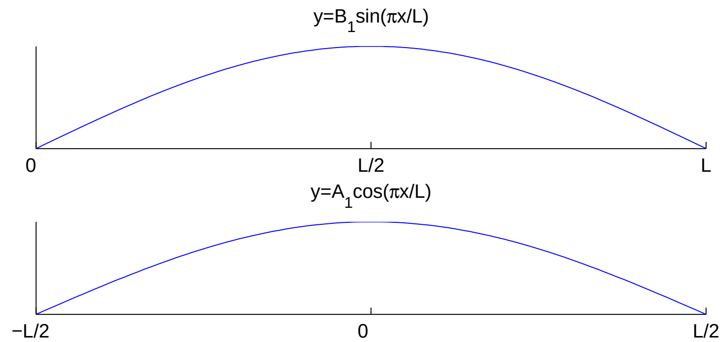
A graph of $y(x)$ for values $x < 0$ and $x > L$ and $n = 1 \rightarrow 999$ is shown below.



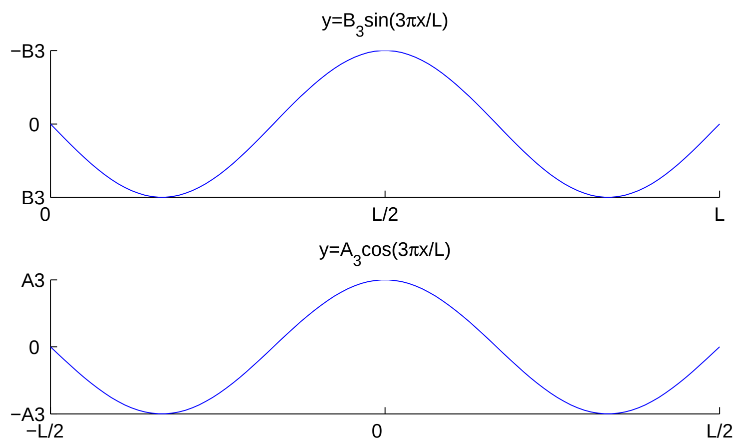
Note that the spatial period of this function is $2L$ and the mean value over this period is zero. Alternatively, we could have shifted the function so that the peak was at $x = 0$ and expanded in terms of cosines over a spatial period of 2L. All functions would then be even. Since all we are doing is shifting the function by $L/2$ we expect that the Fourier coefficients of the sine expansion, B_n, are equal in magnitude to the Fourier coefficients A_n in the cosine expansion

$$y(x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right).$$

It easy to see why the magnitudes of the coefficients of the cosine and sine series must equal. Consider the graphs of the first harmonic for each series.



It is then clear that $A_1 = B_1$. The case for the third harmonics is similar. Remember that $B_3 < 0$.



Then, $A_3 = -B_3$. We thus have

$$A_1 = B_1 \quad A_3 = -B_3 \quad A_5 = B_5 \quad A_7 = -B_7 \quad \ldots$$

Alternatively, we could have computed the Fourier expansion where the spatial wavelength is L. In that case, the decomposition of the function (peak at $x = 0$)

$$y(x) = \begin{cases} \frac{2h}{L}x + h & \text{if } -L/2 \leq x < 0 \\ -\frac{2h}{L}x + h & \text{if } 0 \leq x \leq L/2, \end{cases}$$

would have the form

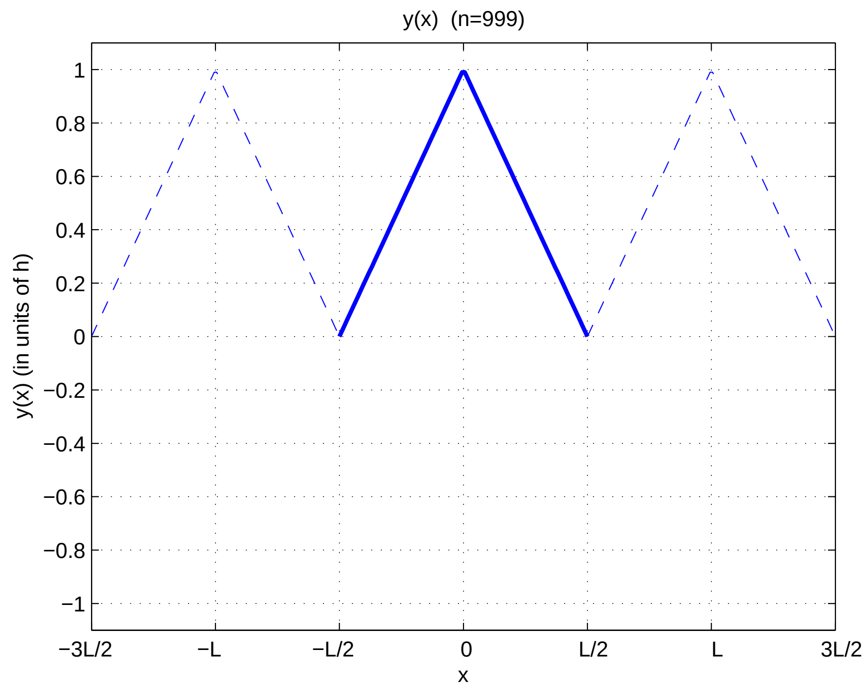
$$y(x) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{2n\pi}{L}x\right).$$

Convince yourself that sine terms are not allowed in this particular (even) Fourier decomposition. The Fourier expansion, in this case, would be

$$y(x) = \frac{h}{2} + \sum_{n=1}^{\infty} \frac{2h}{n^2\pi^2}\left(1 - \cos n\pi\right)\cos\left(\frac{2n\pi}{L}x\right).$$

Note that the constant term $h/2$ is the average value of $y(x)$ over one spatial period. This constant comes from the $n = 0$ term.

The graph of this Fourier expansion is shown below.



This expansion now is even, has a non-zero mean average ($h/2$) and a spatial period L.

Since this Fourier decomposition gives the shape of the original function in the interval $[-L/2, L/2]$, it is a correct mathematical solution. In part (c), however, we will see that this decomposition is not physically correct if we let the string evolve in time.

(b) We know how sinusoids evolve in time. For example, the sinusoid $y(x) = A\sin(kx)$ evolves as $y(x,t) = A\sin(kx)\cos(\omega t + \phi_t)$, where ω is the frequency of oscillations given by the dispersion relation and ϕ_t is the temporal phase of the oscillations. The initial condition $y(x,0) = y(x)$ requires $\phi_t = 0$. Each Fourier component of the string shape $B_n \sin(k_n x)$ will evolve as $B_n \sin(k_n x)\cos(\omega_n t)$, where $\omega_n = k_n v = n\pi\sqrt{T/\mu}/L$. The string shape then evolves as

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi}{L}vt\right).$$

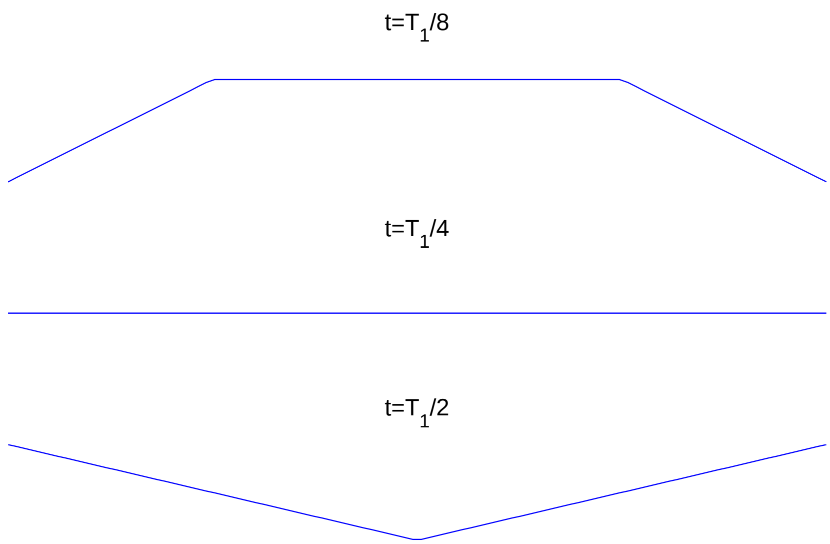
where $v = \sqrt{T/\mu}$ is the speed of propagation.

Could we also have said that the shape of the string evolves as

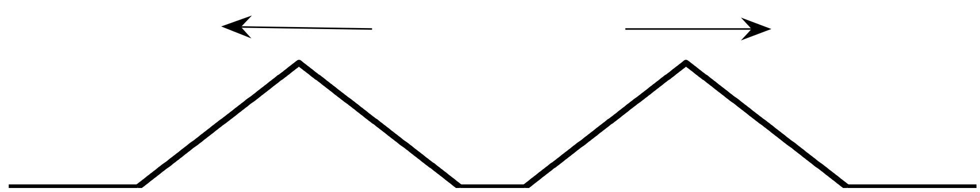
$$y(x,t) = \frac{h}{2} + \sum_{n=1}^{\infty} \frac{2h}{n^2\pi^2} (1 - \cos n\pi) \cos\left(\frac{2n\pi}{L}x\right) \cos\left(\frac{2n\pi}{L}vt\right) \quad ?$$

The answer is NO! Try it, you will notice that at $t = T_1/4$ the entire string is at position $h/2$ (the ends are no longer fixed).

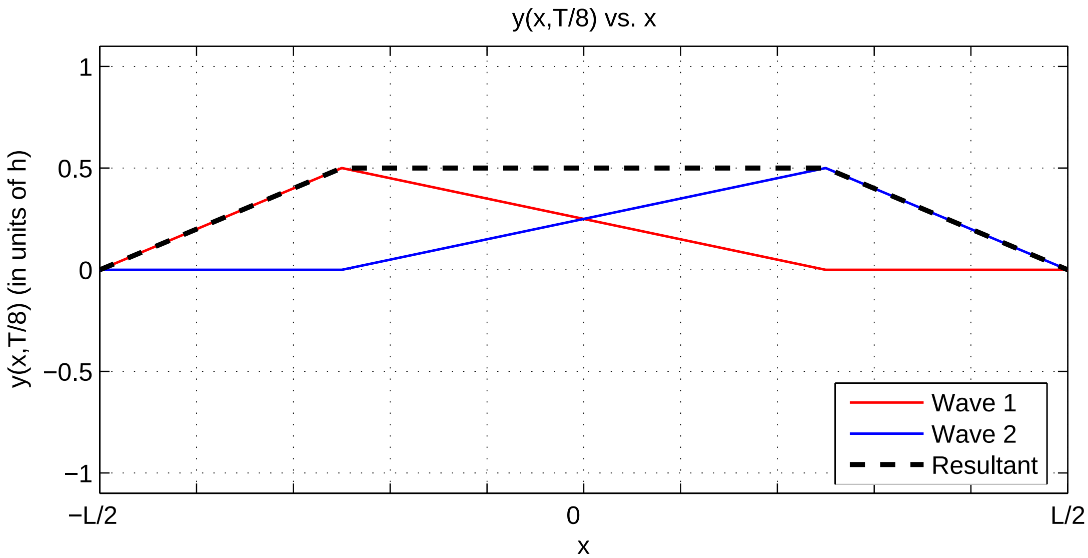
Below are the superpositions of the Fourier standing waves for $t = T_1/8, T_1/4$ and $T_1/2$ ($n = 1 \rightarrow 999$).



(c) There is an alternative way of thinking about the time evolution. The moment you release the string, one triangle (height $h/2$) will travel to the right and the other to the left.



The boundary condition $y(\pm L, t) = 0$ must hold at all times. The graph on the next page shows the travelling waves and the resultant at $t = T/8$. Recall that fixed string ends imply a reflection coefficient of -1. Hence, incident waves flip at the ends of the string. Initially, Wave 1 travels to the left and Wave 2 travels to the right. Notice that 999/2 evolving standing waves and the 2 traveling waves give results that are indistinguishable.



Problem 5.5 Fourier series

The most generic Fourier expansion is

$$y(x) = \sum_{n=0}^{\infty} A_n \cos\left(k_n x + \phi_n\right).$$

The functions in this problem have the boundary conditions $y(0,t) = y(L,t) = 0$, which imply $\phi_n = \pi/2$ and $k_n = \pi n/L$. Hence,

$$y(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right).$$

Note that the sum now starts from $n = 0$ rather than $n = 1$. The $n = 0$ term equals zero so it does not contribute to the sum. The Fourier coefficients, A_n, can be bound by multiplying the latter expression by $\sin\left(k_m x\right)$ and integrating:

$$\begin{aligned}
\int_0^L \sin\left(k_m x\right) y(x)\, dx &= \int_0^L \sin\left(k_m x\right) \sum_{n=1}^{\infty} A_n \sin\left(k_n x\right)\, dx \\
&= \sum_{n=1}^{\infty} A_n \int_0^L \sin\left(k_n x\right) \sin\left(k_m x\right)\, dx \\
&= A_m \frac{L}{2} \\
\Rightarrow A_m &= \frac{2}{L} \int_0^L y(x) \sin\left(\frac{n\pi}{L}\right)\, dx
\end{aligned}$$

(a) The function is

$$y(x) = Ax(1-x)$$

From our discussion above,

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi}{L}x\right)\,dx \\
&= \frac{2}{L}\int_0^L Ax(1-x)\sin\left(\frac{n\pi}{L}x\right)\,dx \\
&= \frac{2AL^2}{\pi^3 n^3}\left(2-\underbrace{2\cos n\pi}_{\text{+1 n even -1 n odd}} -n\pi\underbrace{\sin n\pi}_{=0\,\forall n}\right) \\
\Rightarrow A_n &= \frac{8AL^2}{\pi^3}\frac{1}{n^3} \quad n=1,3,5,7\ldots
\end{aligned}$$

(b) The Fourier expansion of a trigonometric function is itself. By inspection, the Fourier coefficients are

$$A_n = \begin{cases} A & \text{if } n=1 \\ 0 & \text{if } n\neq 0. \end{cases}$$

More formally,

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L A\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\,dx \\
&= \begin{cases} A & \text{if } n=1 \\ 0 & \text{if } n\neq 0. \end{cases}
\end{aligned}$$

(c) The function is

$$y(x) = \begin{cases} A\sin\left(\frac{2\pi}{L}x\right) & \text{if } 0\leq x<L/2 \\ 0 & \text{if } L/2\leq x\leq L. \end{cases}$$

Hence, the Fourier coefficients are

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi}{L}\right)\,dx \\
&= \frac{2A}{L}\int_0^{L/2} \sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\,dx \\
&= -\frac{4A}{\pi}\frac{1}{n^2-4}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

We now must be careful because A_n is ill-defined at $n=2$. We can evaluate A_2 using L'Hopital's rule:

$$\begin{aligned}
A_2 &= -\frac{4A}{\pi}\left.\frac{\frac{\pi}{2}\cos\left(\frac{n\pi}{2}\right)}{2n}\right|_{n=2} \\
&= \frac{A}{2}.
\end{aligned}$$

When n is even (except $n=2$), $A_n=0$. Hence,

$$A_n = \begin{cases} 0 & \text{if } n=4,6,8\ldots \\ \frac{A}{2} & \text{if } n=2 \\ \frac{4A}{\pi}\frac{\sin(n\pi/2)}{4-n^2} & \text{if } n=1,3,5,7\ldots \end{cases}$$

Problem 5.6 — Pianos can talk back

(a) The sounds that you make are a superposition of different frequencies. Each string inside the piano will respond to its harmonics. Hence, the sound of your voice will be broken down into frequencies and selected frequencies will be played back by the piano. In this way, the piano is performing a Fourier analysis of your sound.

The piano need not be in tune, it needs only to possess enough components to make your sound recognizable.

(b) The ratios of the harmonic frequencies of the strings will not be exactly $1 : 2 : 3 \ldots$ because the piano is not tuned that way (see problem 5.1). In addition, the oscillations will not be in phase because of the difference in travel times of your sound to the strings (about 1 meter in 3 msec). In 3 msec the 330 Hz string will perform one complete oscillation; the 1000 Hz will make 3 oscillations, etc.

(c) Apparently, phase in unimportant.

(d) We cannot explain this. But it is the way our brains work. Perhaps evolution did not discover any survival value in keeping the phase.