Non-negativity of the γ-vector for *3*-Dimensional Polytopes

by:
Robert B. Zima

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Honors Committee

Dr. Kristin Camenga

Dr. David Perkins

Dr. Mark LaCelle - Peterson

Abstract:

The general focus of this honors thesis is to further understand the properties of solid angles of convex polytopes as recorded in the γ-vector. It is conjectured that all the entries of the *γ*-vector are non-negative for polytopes in any dimension. Since entries of the *γ*-vector are found by adding and subtracting the measures of angles in different dimensions, this is non-trivial. This project will make progress towards proving the conjecture for *3*-dimensional polytopes.

This project was approached through the concepts and methods of geometric combinatorics. Dr. Kristin Camenga has proved that the γ-vector is non-negative for all *3*-simplices and *4*-simplices, where a simplex is defined as a *d*-dimensional object with *(d + 1)* vertices. It is also known that the *γ*-vector is non-negative for all *2*-dimensional polytopes. Since we already know this to be true, we will look at connections between *γ*-vectors of prisms and pyramids and to the *γ*-vectors of their *2*-dimensional bases. By subdividing pyramids and prisms into simplices, known results were applied to make deductions for pyramids and prisms.

Through this honors project, we proved that all *3*-dimensional pyramids have non-negative *γ*-vectors. We also showed $\gamma_1(P) + \gamma_2(P) = \frac{n-1}{2}$ for a pyramid over an *n*-gon. For prisms, we were able to demonstrate the relationship between the angle sums of a prism and the angle sums of the simplices in a subdivision of the prism. Furthermore, $\gamma_1(P) + \gamma_2(P) = \frac{n}{2}$ for a prism over an *n*-gon.

CHAPTER 1
Polytopes

Convex polytopes are geometric objects, which we are familiar with in *3*-dimensions. While *3*-dimensional models are exceptionally helpful for understanding the general makeup and properties of convex polytopes, the concepts are generalized to higher dimensions. Many of the theorems regarding *3*-dimensional convex polytopes do not apply to polytopes in higher dimensions.

Constructing Polytopes

Polytopes of all dimensions are constructed in two ways. The first construction finds the convex hull for a finite set of points in any dimension. The second is constructed by the intersection of half-spaces. While each is different in its constructive approach, the same set of objects can be constructed.

Objects are considered convex if for any two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}$, the line segment between $\boldsymbol{x}$ and $\boldsymbol{y}$ is also contained in $\mathbf{C}$. Symbolically, a point set $K \subseteq \mathbf{R}^d$ is convex if with any two points $x, y \in K$ it also contains the straight line segment $[x, y] = \{\lambda x + (1 - \lambda) y : 0 \leq \lambda \leq 1\}$ between them. For a set of points, the convex hull is defined as the smallest convex set containing K that can be constructed as the intersection of all convex sets that contain K.

Figure 1.1: To the left, a non-convex polygon & to the right, a convex polygon

Half-spaces in $\mathbf{R}^d$ are defined as the set of all points in and on one side of a $d - 1$ space, which occupies half of $\mathbf{R}^d$. In a *2*-dimensional context, the intersection of half-spaces defined by *1*-dimensional lines forms a polygon. The set of all points on one side of plane in *3*-space is also a half-space. By intersecting these *3*-dimensional spaces, a *3*-dimensional polytope is formed.

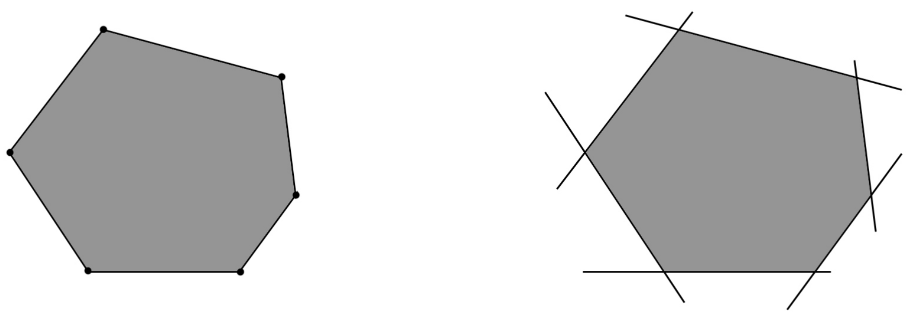

Figure 1.2: The convex hull and intersection of halfspaces forms a convex polytope

A d-simplex is a special polytope defined as the convex hull of any $d + 1$ affinely independent points in some $\mathbf{R}^n$ ($n \geq d$). Affine independence implies that a d-simplex is a polytope of full dimension d. A *2*-dimensional simplex is a triangle. A *3*-dimensional simplex is called a tetrahedron. Simplices can be used as building blocks for more complex polytopes.

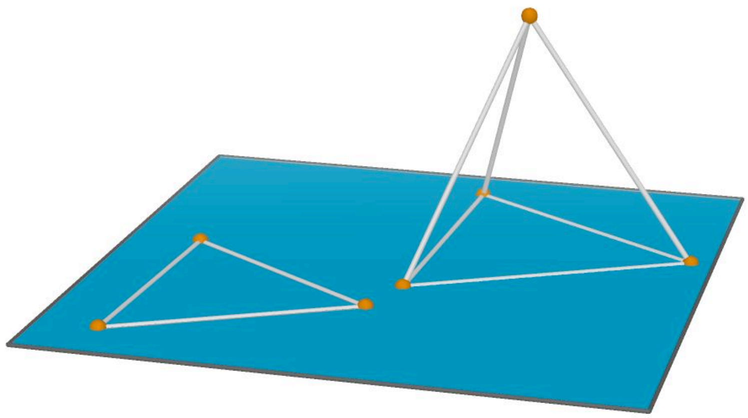

Figure 1.3: Connecting all vertices in a 2-simplex to an apex forms a 3-simplex

Faces

Each polytope *P* consists of lower dimensional faces. In *3*-dimensional polyhedra, we commonly use such terms as vertices, edges, and faces and consider all of them as faces of the polytope. In the context of general polytopes, vertices are referred to as *0*-faces; edges are referred to *1*-faces; faces are called *2*-faces; the whole polytope, or the cell, will be called *3*-faces; and *i*-dimensional faces will be called *i*-faces. Faces, not only *2*-faces, generally define the boundary and structure of any *d*-dimensional polytope.

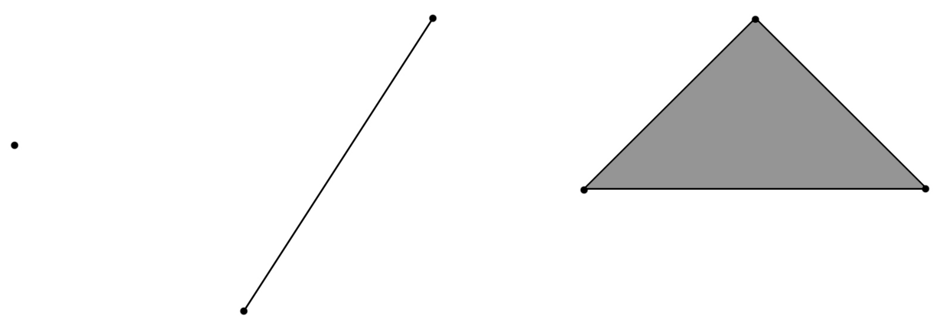

Figure 1.4: Vertex, edge, face

We will denote the number of i-faces of P as $f_i(P)$. Therefore, in a 3-dimensional context, $f_0(P)$ counts the number of vertices, $f_1(P)$ counts the number of edges, $f_2(P)$ counts the number of faces, and $f_3(P)$ counts the number of cells. When we consider the standard cube, it can be seen that $f_0(P) = 8$, $f_1(P) = 12$, $f_2(P) = 6$, and $f_3(P) = 1$.

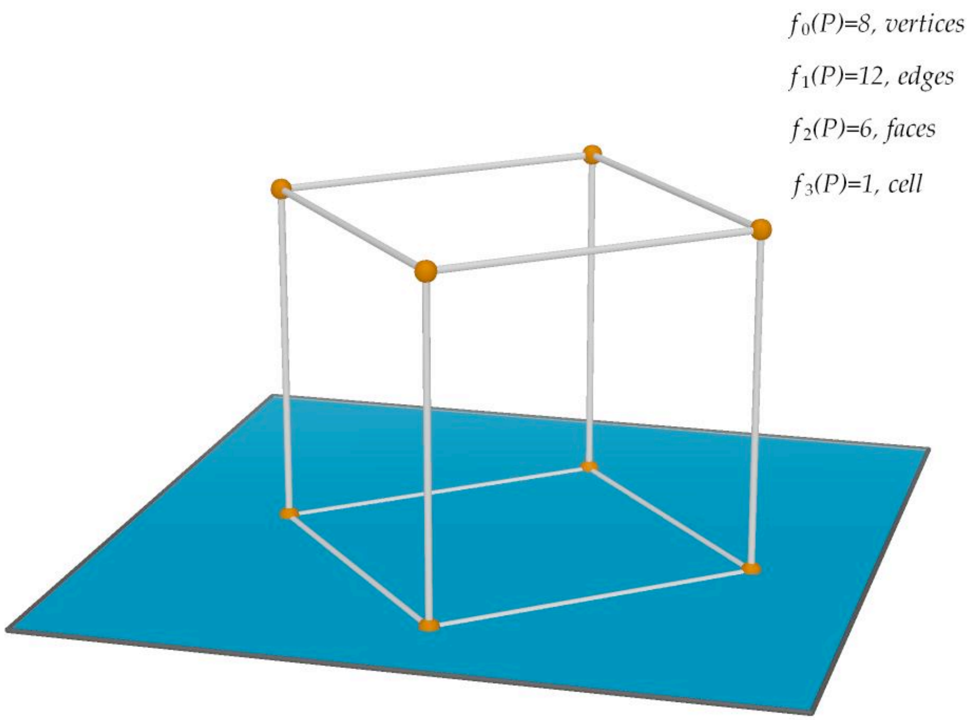

Figure 1.5: Square prism with $f_i(P)$ denoted

The f-vector is defined as $f(P) = (f_0(P), f_1(P),\dots, f_d(P))$. Notice that when we alternately add and subtract the terms in the f-vector for a 3-dimensional cube, we get 1, i.e. $8 - 12 + 6 - 1 = 1$. This generalizes to all polytopes at d-dimensions and is known as the Euler relation. That is:

$$\sum_{i=0}^{d}(-1)^i f_i(P) = 1 \text{ for any } d\text{-polytope } P.$$

The h-vector, which is defined as $h(P) = (h_0(P), h_1(P), \ldots, h_d(P))$, where

$$h_i(P) = \sum_{j=0}^{i}(-1)^{i-j}\binom{d-j}{d-i} f_{j-1}(P)$$

is a linear transformation of the f-vector. Within this formula, $f_{-1}(P) = 1$ by convention. Work with polytopes has led to understanding of the h-vector and its physical meaning.

Any d-polytope can be subdivided into simplices. Consider a 2-dimensional n-gon, a polygon with n sides. Any n-gon can be subdivided into triangles, or 2-dimensional simplices. This is analogous to subdividing polytopes in 3-dimensions, which will be discussed in more detail later.

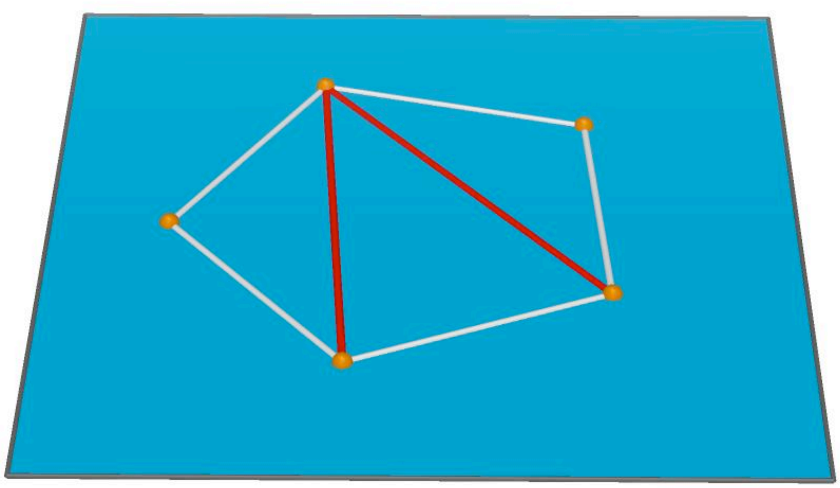

Figure 1.6: Pentagon subdivided into 3 triangles

There exist a variety of polytopes, yet this honors project deals primarily with two specific types: pyramids and prisms, which are defined in Chapter 3. The *f*-vectors of all pyramids and prisms will span all the possible *f*-vectors of polytopes. Therefore, these polytopes can be thought of as building blocks for all polytopes, so results for these classes of polytopes may lead to results for general polytopes.

Chapter 2

Angle Sums of Polytopes

Last chapter primarily dealt with understanding the basic construction of 3-dimensional polytopes and the faces that comprise the polytope. In this chapter, we will define the α-vector and γ-vector for a polytope, which are the main focus of this study.

The α-vector

The angle sums of any *d*-polytope *P* are analogous to entries of the *f*-vectors. We denote the i^{th} angle sum as $\alpha_i(P)$, which is the sum of the interior angles at every *i*-face of *P*. The interior angle at an *i*-face *F* is the fraction of a *d*-dimensional ball centered at an interior point of *F*. These *d*-dimensional balls are small enough so that they do not intersect faces of lower dimension.

Then we define the α-vector as $\alpha(P) = (\alpha_0(P), \alpha_1(P), \ldots, \alpha_d(P))$.

Recall our example of the cube. Consider a vertex. When we center a small *3*-dimensional ball at the vertex, there is precisely $\frac{1}{8}$ of the ball contained within the cube, so the interior angle is $\frac{1}{8}$. When we consider an edge, $\frac{1}{4}$ of the ball is contained within the cube, so the interior angle is $\frac{1}{4}$. When placing a ball on a face, $\frac{1}{2}$ of the ball is contained within the cube, so the interior angle is $\frac{1}{2}$. Finally, the cell will contain the entire ball for an interior angle of *1*.

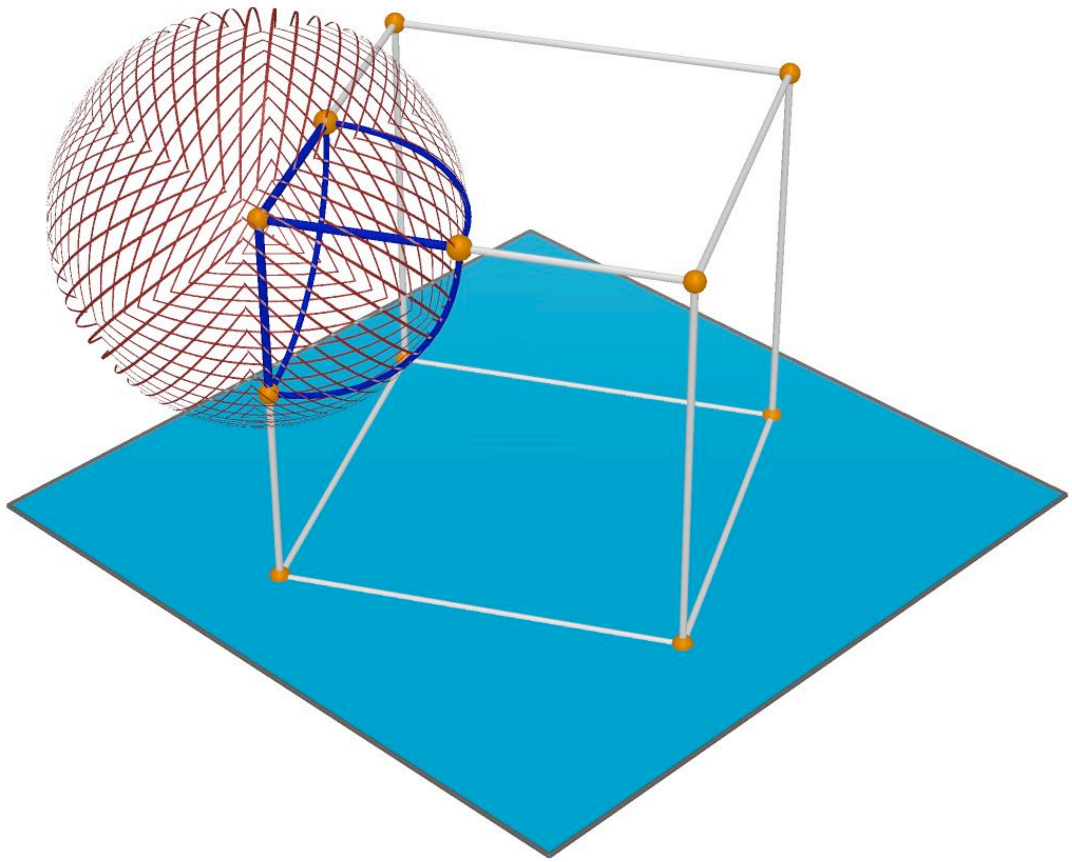

Figure 2.1: 3-dimensional ball measuring interior angle at a vertex

When dealing with angle sums we need to add the interior angles of every face in a given dimension. When the angles of a polytope are equivalent at every *i*-face, we only need to multiply the interior angles at an *i*-face by the number of *i*-faces or $f_i(P)$. In this example, the angles at different faces of a given dimension are the same because the cube is symmetrical, which is why we are multiplying the interior angle at every face by its corresponding $f_i(P)$. Therefore in the cube,

$(8)\cdot\frac{1}{8} = 1 = \alpha_0(P)$,

$(12)\cdot\frac{1}{4} = 3 = \alpha_1(P)$,

$(6)\cdot\frac{1}{2} = 3 = \alpha_2(P),$

$(1)\cdot 1 = 1 = \alpha_3(P).$

So the α-vector for the cube is (1, 3, 3, 1).

Much like the Euler relation, $\sum_{i=0}^{d}(-1)^i f_i(P) = 1$, the Gram relation states that

$$\sum_{i=0}^{d}(-1)^i \alpha_i(P) = 0 \text{ for any } d\text{-polytope } P.$$

We can make a number of general observations about angle sums. For d-polytopes, $\alpha_d(P) = 1$. When we consider higher dimensional polytopes, $\alpha_d(P)$ is found by placing a d-dimensional ball within the polytope, which will always yield a result of 1. In 3-dimensions, $\alpha_3(T) = 1$. Furthermore, $\alpha_{d-1}(P) = \frac{f_{d-1}(P)}{2}$. The same sort of argument as above applies in this instance, where $f_{d-1}(P)$ counts the number of $d - 1$ faces, and the interior angle at each $d - 1$ face contains exactly half of the d-dimensional ball.

Consider a 3-simplex, call it T. Using the above results, $\alpha_2(T) = \frac{f_2(P)}{2} = 2$ and because $\alpha_3(T) = 1$, we can substitute these values in the Gram relation. Thus, $\alpha_0(T) - \alpha_1(T) + 2 - 1 = 0$, which implies that $\alpha_0(T) + 1 = \alpha_1(T)$. It is also known that $\alpha_0(T) > ½$ (Dr. Camenga, unpublished).

The γ-vector

Let us then define the γ-vector as

$\gamma(P) = (\gamma_0(P), \gamma_1(P), \ldots, \gamma_d(P))$,

where

$$\gamma_i(P) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} \alpha_{j-1}(P).$$

This is analogous to the *h*-vector in that we are taking a linear transformation of the α-vector to find each $\gamma_i(P)$. Unlike the *h*-vector, there is no known physical interpretation. Within this formula, we will use $\alpha_{-1}(P) = 0$ by convention.

The γ-vector entries for 3-dimensional polytopes are:

$$\gamma_0(P) = \sum_{j=0}^{0} (-1)^{0-j} \binom{3-j}{3} \alpha_{j-1}(P) = \binom{3}{3} \alpha_{-1} = 0.$$

$$\gamma_1(P) = \sum_{j=0}^{1} (-1)^{1-j} \binom{3-j}{2} \alpha_{j-1}(P) = -\binom{3}{2} \alpha_{-1}(P) + \binom{2}{2} \alpha_0(P) = \alpha_0(P).$$

$$\gamma_2(P) = \sum_{j=0}^{2} (-1)^{2-j} \binom{3-j}{1} \alpha_{j-1}(P) = \binom{3}{1} \alpha_{-1}(P) - \binom{2}{1} \alpha_0(P) + \binom{1}{1} \alpha_1(P) = \alpha_1(P) - 2\alpha_0(P).$$

$$\gamma_3(P) = \sum_{j=0}^{3} (-1)^{3-j} \binom{3-j}{0} \alpha_{j-1}(P) = -\binom{3}{0} \alpha_{-1}(P) + \binom{2}{0} \alpha_0(P) - \binom{1}{0} \alpha_1(P) + \binom{0}{0} \alpha_2(P)$$

$$= \alpha_0(P) - \alpha_1(P) + \alpha_2(P) = 1.$$

It follows from the Gram relation in 3-dimensions that $\gamma_3(P) = 1$. The sum $\alpha_0(P) - \alpha_1(P) + \alpha_2(P) - \alpha_3(P) = 0$ can be rewritten in the form $\alpha_0(P) - \alpha_1(P) + \alpha_2(P) = \alpha_3(P) = 1$.

Currently, we know that for the 3-simplex and 4-simplex, the γ-vector is non-negative (Dr. Camenga, unpublished). We will show the γ-vector is non-negative for any 3-dimensional pyramid.

Chapter 3

Computer Software

This particular study incorporated the uses of two main pieces of technology to formulate and investigate conjectures about the γ-vector. Cabri 3D is a computer software program designed to aid in visualization of polytopes and provide coordinates of the vertices. A new program, specifically designed for this honors project, is able to compute the α-vector and γ-vector for 3-dimensional polytopes

Visualizations

Cabri 3D is a software program that allows us to visually represent 3-dimensional polytopes. Visualization of 3-dimensional polytopes can become exceptionally difficult. By using this software, we are able to easily visualize the object with which we are working from multiple perspectives. By using this program, we can also subdivide polytopes into a set of connected 3-simplices. In doing this, we hope to apply our knowledge of γ-vectors for 3-simplices to more generalized polytopes of prisms and pyramids.

Pyramids

The process of forming a pyramid is fairly straightforward. We first construct an n-gon. A pyramid is constructed by adding a vertex, also called an apex, off the plane of the polygonal base.

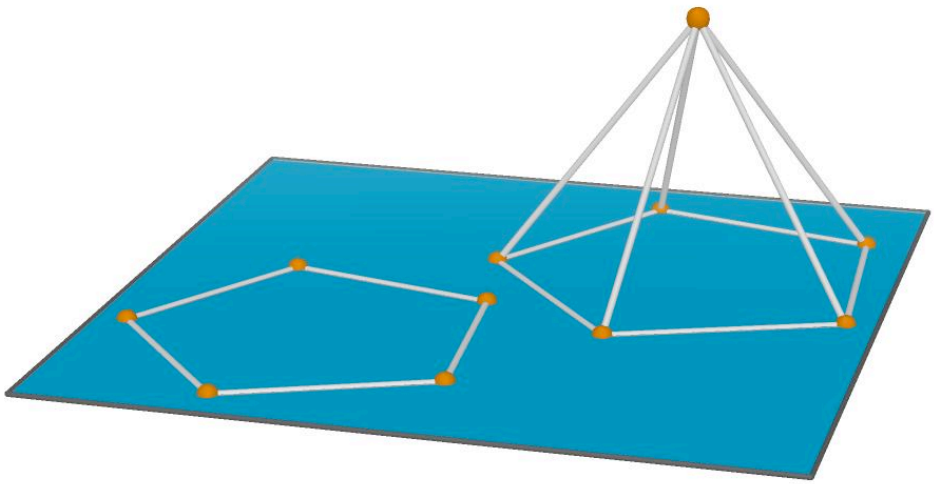

Figure 3.1: Formation of a pentagonal pyramid from a pentagon

Since we presently know that the γ-vector is non-negative for all *3*-simplices, we proceed by subdividing pyramids and prisms into multiple simplices with connected vertices, edges, and faces. One way to subdivide pyramids into simplices is to form triangles from a vertex in the base. Then, when the apex is added, each triangle forms a simplex with shared vertices, edges, and faces. Precise details will be discussed in Chapter 4.

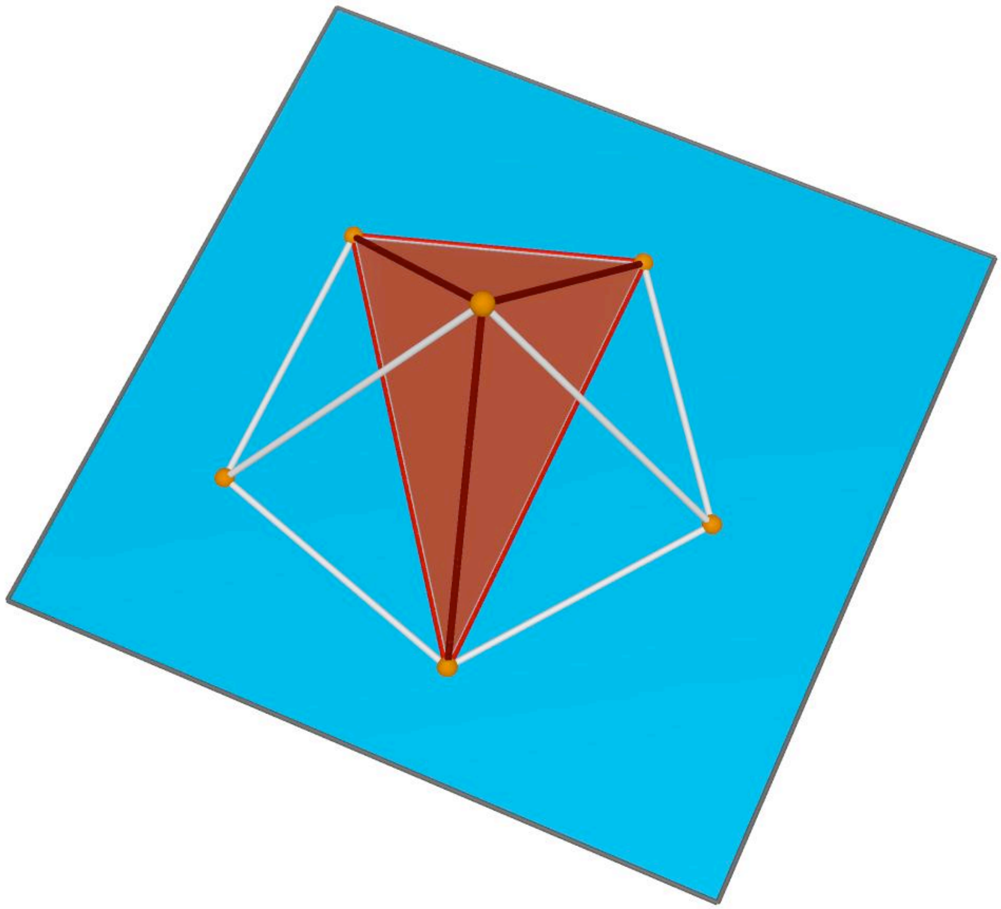

Figure 3.2: Subdivided pentagonal pyramid into 3 simplices

Prisms

Prisms are constructed in a similar manner. First, we form a polygonal base of *n* sides. If we then duplicate the polygon and place it parallel to the plane on which the polygon rests and connect corresponding vertices, a prism is created.

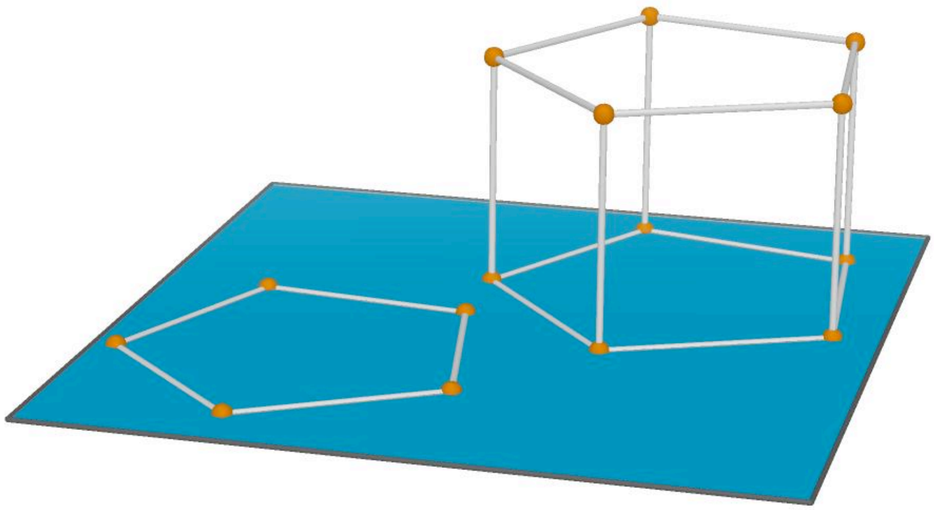

Figure 3.3: Formation of a pentagonal prism from a pentagon

We will consider two ways in which to subdivide prisms into simplices. Place a vertex inside the prism at the centroid. Connect the exterior vertices to the interior vertex, which will form n square pyramids with two n-sided pyramids, each of which have the centroid vertex as the apex. Square pyramids can be subdivided into two simplices. The two n-sided pyramids will then have multiple simplices within them as well. Another method by which prisms can be split into simplices is to subdivide both polygonal bases into triangles. The result is multiple triangular prisms. Then each triangular prism can be subdivided into three simplices. The precise details of these subdivisions will be discussed later.

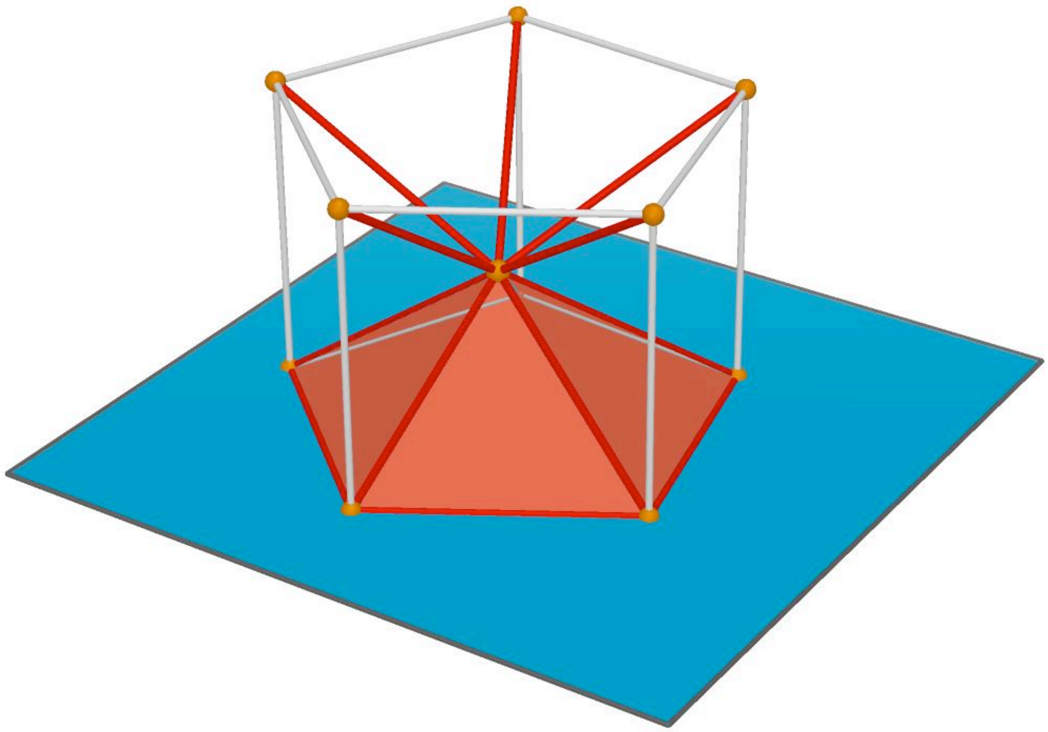

Figure 3.4: Method of subdividing prisms into simplices

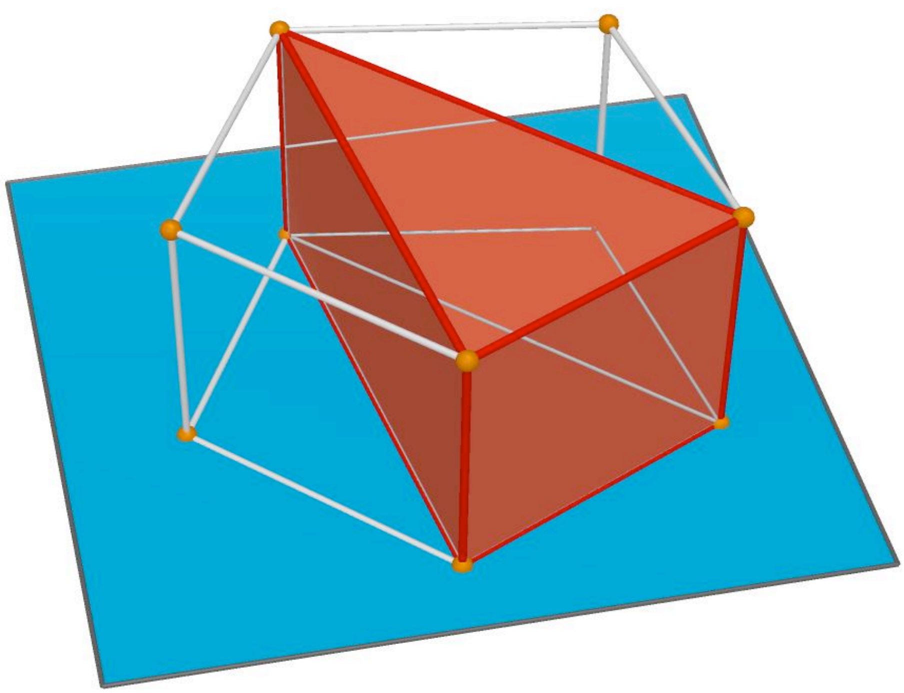

Figure 3.5: Method of subdividing prisms into simplices

Computational Software

Willard Frutiger and Jeremy Collins designed another piece of computer software, specifically for the purpose of calculating α and γ vectors. First, using Cabri 3D as the constructive agent, we will form a prism or pyramid and find specific coordinates for each vertex. Cabri 3D has the ability to provide ***x, y,*** and ***z*** coordinates for any vertex of a polytope. We then take these ***x, y,*** and ***z*** values and insert them into a previously designed template. From the given vertices, the program determines possible edges, faces, and cells as *1*-dimensional, *2*-dimensional, and *3*-dimensional subspaces.

```
# 3-simplex

vect:{p1, (1.8, 2.7, 0)}
vect:{p2, (-4.2, -0.9, 0)}
vect:{p3, (1.9, -4.3, 0)}
vect:{p4, (-0.2, -0.9, 3.2)}


macro:{0; step; original; 50; [(p1>p1), (p2>p2), (p3>p3), (p4>(-3.4, 0.9, 6))]}
macro:{1; step; current; 50; [(p1>p1), (p2>p2), (p3>p3), (p4>(-3.4, 0.9, 4))]}
macro:{2; step; current; 50; [(p1>p1), (p2>p2), (p3>p3), (p4>(-3.4, 0.9, 2))]}

@dimension 3
```

Figure 3.6: Program input for a 3-simplex

In order to calculate the α-vector, the program uses multiple methods involving linear algebra. Normal vectors are computed for each plane of the polytope. A normal vector is perpendicular to a given plane. Using the Gram–Schmidt process, all normal vectors are turned inward. When normal vectors are then scaled to unit vectors, namely $\mathbf{n}_1$ and $\mathbf{n}_2$, we can find the dihedral angles, or interior angles at an edge formed by two planes, by using dot-products. The dot-product is an operation which takes two vectors and returns a scalar quantity. For example,

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \bullet \begin{bmatrix}4\\5\\6\end{bmatrix} = (1)(4) + (2)(5) + (3)(6) = 32.$$

It is well known that the dot-product of two unit vectors is the same as the cosine of the angle between the vectors:

$\cos\theta = \mathbf{n_1} \bullet \mathbf{n_2}$
or $\arccos(\mathbf{n_1} \bullet \mathbf{n_2}) = \theta$.

Since this method gives us an angle out of a full circle of 360^o, we must divide the angle $\theta = \arccos(\mathbf{n_1} \bullet \mathbf{n_2})$ by 360^o to have our angle in terms of a ratio, which defines the interior angle at the edge. Then, $\alpha_1(P)$ is the sum of these interior angles, $\alpha_2(P) = \frac{f_2(P)}{2}$, and $\alpha_3(P) = 1$. We find $\alpha_0(P) = \alpha_1(P) - \alpha_2(P) + \alpha_3(P)$ using the Gram relation.

This process can be repeated multiple times within the program by performing transformations to the vertices. For example, if we have a symmetric square pyramid, we can move the apex in multiple directions and calculate the angles of the resulting polytopes. As seen in Chapter 2, we can then compute each $\gamma_i(P)$. The results of these trials may lead to a variety of conjectures.

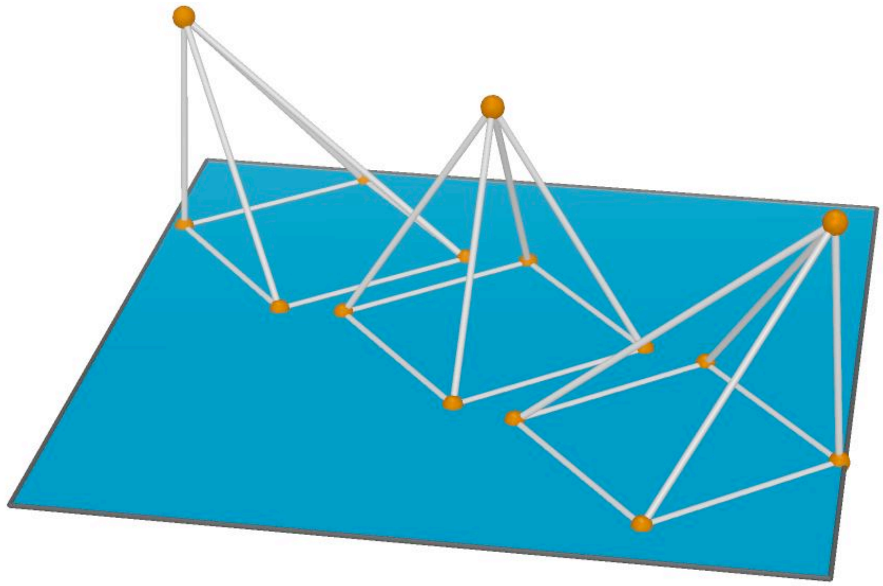

Figure 3.7: Transformation of a square pyramid by moving the apex

The computer program is currently able to calculate α-vectors of *3*-dimensional polytopes; however work is progressing on the program to compute angle sums for higher dimensional polytopes. Using the software program, we were able to make a variety of conjectures based on data returned.

Chapter 4

Pyramid

Last chapter included visual representations of polytopes using Cabri 3D and the process through which we subdivide pyramids into simplices. In this chapter, we will use the subdivision process to prove that the γ-vector of pyramids is non-negative.

Subdivision of polytopes into simplices requires that interior faces are added to the polytope. This means that when summing angles over the faces in the subdivision, the angles at these added faces will be in addition to the angle sums of the original polytope. Since the γ-vector is computed from the α-vector, we will show that the γ-vector is non-negative by computing $\alpha_i(P)$ based on the angle sums of the simplices in the subdivision.

Note: The arguments will be illustrated using pentagonal pyramids, although the arguments apply to all pyramids over a polygon.

Pyramid Subdivision

Consider a pyramid PQ. Let n be the number of vertices of the base polygon. When we select a vertex at which all polytopes in the subdivision will meet, we see that there are $n - 3$ interior edges in the base, forming $n - 2$ triangles. Thus, connecting each of these newly created triangles to the apex forms $n - 2$ simplices, which we will name $T_1, T_2, \ldots, T_{n-2}$.

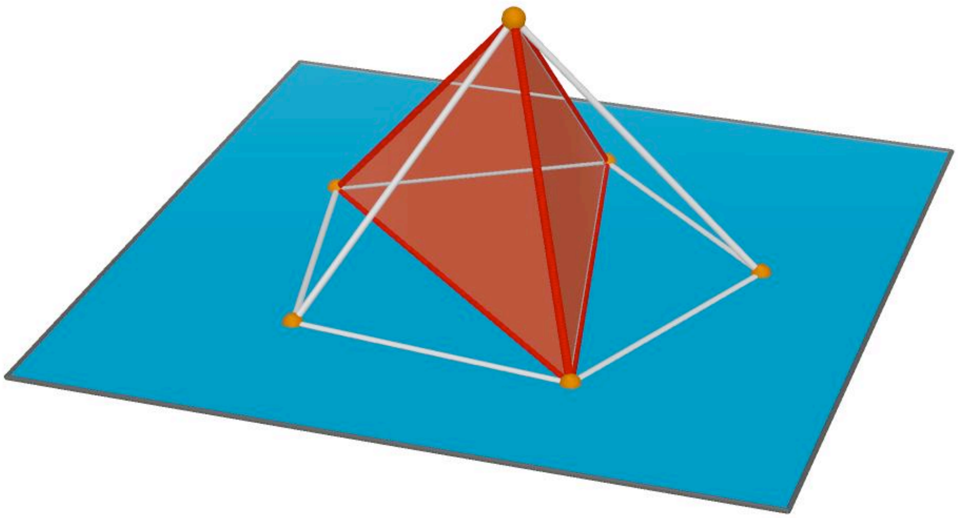

Figure 4.1: Three simplices formed when a pentagonal prism is subdivided

Each of these simplices, T_i, contributes 1 to $\sum_{i=1}^{n-2}\alpha_3(T_i) = n - 2$ while the original simplex only had $\alpha_3(PQ) = 1$ for a difference of $n - 3$. Thus, we subtract $n - 3$ from $\sum_{i=1}^{n-2}\alpha_3(T_i)$ when subdividing into simplices, since the original pyramid has the property $\alpha_3 = 1$. Thus

$$\alpha_3(PQ) = \sum_{i=1}^{n-2}\alpha_3(T_i) - (n-3).$$

Recall that α_2 will always be the number of faces of a particular polytope multiplied by ½. When there are n vertices, there will always be n faces adjacent to the apex and one base, each with an angle of ½, so $\alpha_2(PQ) = \frac{n+1}{2}$ for any pyramid. The subdivision of the base into $n - 2$ triangles results in an additional $n - 3$ faces in the base, which contribute an extra $\frac{n-3}{2}$ to $\sum_{i=1}^{n-2} \alpha_2(T_i)$. There are also $n - 3$ faces formed inside the polytope when subdivision occurs. Since the interior triangles are the boundary between two simplices in the subdivision, the angle will count ½ for both simplices, for a total of $\frac{2(n-3)}{2}$ at the interior faces of the subdivision. The outer faces between the base and the apex will continue to contribute $\frac{n}{2}$ to $\sum_{i=1}^{n-2} \alpha_2(T_i)$. Thus, the only subtractions necessary are the angles of the newly formed triangles in the base and the interior triangles that extend to the apex.

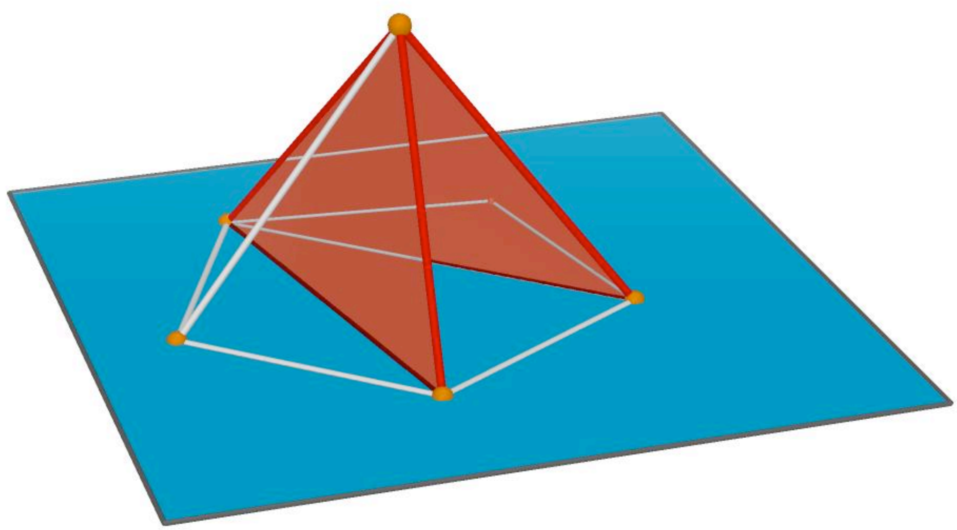

Figure 4.2: Two interior faces formed

So, $\frac{2(n-3)}{2}+\frac{n-3}{2}=\frac{3(n-3)}{2}$ is subtracted from $\sum_{i=1}^{n-2}\alpha_2(T_i)$ to get $\alpha_2(T_i)$, where $\frac{2(n-3)}{2}$ measures the angles at the additional interior faces and $\frac{n-3}{2}$ measures the angles at the additional exterior faces . Thus

$$\alpha_2(PQ)=\sum_{i=1}^{n-2}\alpha_2(T_i)-\frac{3(n-3)}{2}.$$

The interior angle at an edge varies for different edges. When we considered α_2, we knew that the angles at each face would always be ½; however the angle at one edge may be entirely different from that of another edge. Yet when we subdivide a pyramid with $n-3$ subdivisions, the only edges created are those in the base. Each new interior edge will have a total angle of ½. The angle of an interior edge will be split by an interior triangle, and the sum of the two angles will remain ½. The same argument applies for exterior edges in the original pyramid. The interior angles in the two simplices that meet at an edge will sum to the original interior angle of a particular exterior edge. Thus, we do not need to be concerned with how they are divided, but only note that the sum is constant, so the sum of the interior angles at an edge does not vary.

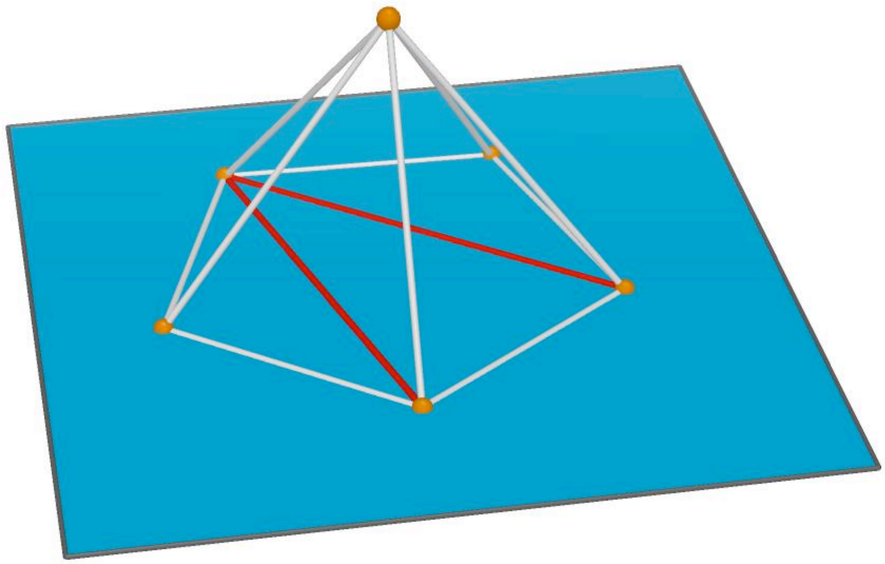

Figure 4.3: Two edges formed

Thus, $\frac{n-3}{2}$, which is the measure of angles at the new edges, is subtracted from $\sum_{i=1}^{n-2}\alpha_1(T_i)$ to get $\alpha_i(PQ)$. So

$$\alpha_1(PQ) = \sum_{i=1}^{n-2}\alpha_1(T_i) - \frac{n-3}{2}.$$

The angle sum at each vertex does not change. When polytopes of the subdivision meet at a vertex, they also subdivide the angle. Thus, the angles in the T_i that meet at a vertex will sum multiple angles to original angle of the vertex. Thus,

$$\alpha_0(PQ) = \sum_{i=1}^{n-2} \alpha_0(T_i).$$

Recall

$$\gamma_0(P) = 0,\ \gamma_1(P) = \alpha_0(P),\ \gamma_2(P) = \alpha_1(P) - 2\alpha_0(P),\ \gamma_3(P) = 1.$$

We can see that $\gamma_0(PQ)$, $\gamma_1(PQ)$, and $\gamma_3(PQ)$ are always non-negative immediately. The angle sum at $\alpha_0(P)$ will always be a positive number, thus $\gamma_1(PQ)$ is positive. Now, we must show that $\gamma_2(PQ)$ is non-negative.

Since we subdivided the pyramid into multiple simplices, it is necessary to sum each angle sum from every simplex in the subdivided polytope. We will use the fact that $\alpha_1(T) = \alpha_0(T) + 1$ to express $\alpha_1(PQ)$ in terms of $\alpha_0(T_i)$ for $i = 1, 2, \ldots, n-2$.

Observe:

$$\alpha_1(PQ) = \sum_{i=1}^{n-2} \alpha_1(T_i) - \frac{n-3}{2}$$

$$= \sum_{i=1}^{n-2} (\alpha_0(T_i) + 1) - \frac{n-3}{2}$$

$$= \sum_{i=1}^{n-2} \alpha_0(T_i) + (n-2) - \frac{n-3}{2}$$

$$= \sum_{i=1}^{n-2} \alpha_0(T_i) + \frac{n-1}{2}.$$

To show $\gamma_2(PQ) \geq 0$, we will substitute the above into the equation for $\gamma_2(PQ)$ along with $\alpha_0(PQ) = \sum_{i=1}^{n-2} \alpha_0(T_i)$.

$$\gamma_2(PQ) = \alpha_1(PQ) - 2\alpha_0(PQ).$$

$$= \sum_{i=1}^{n-2} \alpha_0(T_i) + \frac{n-1}{2} - 2\sum_{i=1}^{n-2} \alpha_0(T_i)$$

$$= \frac{n-1}{2} - \sum_{i=1}^{n-2} \alpha_0(T_i)$$

Recall that $\alpha_0(T_i) < ½$ from Chapter 2. This implies that $\sum_{i=1}^{n-2} \alpha_0(T_i) < \frac{n-2}{2}$. We add every $\alpha_0(T_i)$ for each simplex on the left, whereas on the right we have precisely $n - 2$ simplices times ½.

$$\gamma_2(PQ) = \frac{n-1}{2} - \sum_{i=1}^{n-2} \alpha_0(T_i) > \frac{n-1}{2} - \frac{n-2}{2} = \frac{1}{2}.$$

Therefore, $\gamma_2(PQ) > \frac{1}{2}$.

By the work above, we have proved the following:

Theorem 1:

Since $\gamma_0(PQ)$, $\gamma_1(PQ)$, $\gamma_2(PQ)$, and $\gamma_3(PQ)$ are all positive values in 3-dimensions, the γ-vector is non-negative for all 3-dimensional pyramids.

□

Using the relationships between $\alpha_j(PQ)$ and $\sum_{i=1}^{n-2} \alpha_j(T_i)$ we can also prove the following proposition.

Proposition 1:

For all 3-dimensional pyramids PQ, $\gamma_1(PQ) + \gamma_2(PQ) = \frac{n-1}{2}$, where n represents the number of sides in the base pyramid.

Proof:

Consider each respective $\gamma_i(PQ)$ in terms of $\alpha_i(PQ)$. We have

$$\gamma_1(PQ) + \gamma_2(PQ) = (\alpha_1(PQ) - 2\alpha_0(PQ)) + \alpha_0(PQ) = \alpha_1(PQ) - \alpha_0(PQ).$$

We then have

$$\begin{aligned}\gamma_1(PQ) + \gamma_2(PQ) &= \alpha_1(PQ) - \alpha_0(PQ) \\ &= \sum_{i=1}^{n-2} \alpha_0(T_i) + \frac{n-1}{2} - \sum_{i=1}^{n-2} \alpha_0(T_i) \\ &= \frac{n-1}{2}.\end{aligned}$$

Thus, for all *3*-dimensional pyramids,

$$\gamma_1(PQ) + \gamma_2(PQ) = \frac{n-1}{2}.$$

□

CHAPTER 5
Prism

Prism Subdivision

Again, we will consider the relationships between angles sums of a prism and those of the simplices in the subdivision. Consider a prism, denoted *BQ*. Let *n* be the number of vertices of the base polygon. Recall from Chapter 3 the process of subdividing prisms. As with pyramids, we will subdivide each of our two base polygons from corresponding vertices. Notice that subdivision of the upper base polygon will mirror that of the lower base polygon. Then we look at the prisms made by corresponding triangles. There will always be $n - 2$ triangular prisms formed, each of which can be subdivided into three simplices. Thus, the total number of simplices formed when we subdivide is $3(n - 2)$.

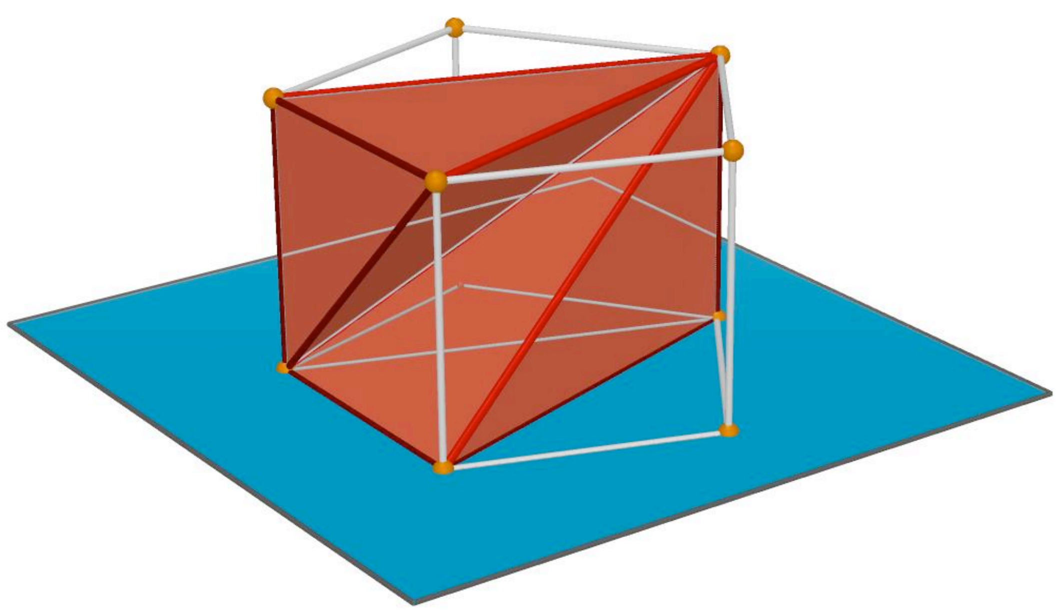

Figure 5.1: Three simplices formed within each triangular prism

Since the original prism has $\alpha_3(BQ) = 1$ and each T_i has $\alpha_3(T_i) = 1$,

$$\alpha_3(BQ) = \sum_{i=1}^{3(n-2)} \alpha_3(T_i) - (3(n-2)-1).$$

When we consider $\alpha_2(BQ)$, the computational subtractions necessary are far more complicated after subdivision than any other case. Here, $\alpha_2(BQ) = \frac{n+2}{2}$ for any prism that has not been subdivided, since there are $n + 2$ faces. When we subdivide the prism into $3(n - 2)$ simplices, we are adding faces between simplices within each triangular prism, between triangular prisms, and on outer faces.

Any interior face contributes ½ to $\alpha_2(T_i)$ of two different simplices for a total of *1*. First, we will consider the interior faces between simplices in each triangular prism formed from subdivision. Since there are three simplices, there will be two interior faces formed within each triangular prism. Thus, we have $2(n - 2)$ interior faces within triangular prisms, since there are $n - 2$ triangular prisms formed from subdivision.

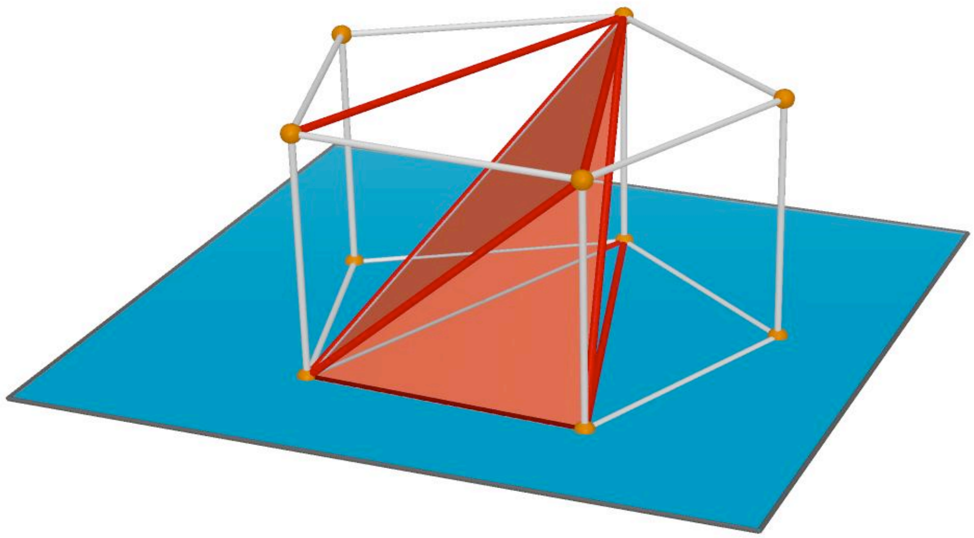

Figure 5.3: Two faces formed within each triangular prism between simplices

Next, we consider faces between triangular prisms. There are two triangular faces on each interior vertical face, for a total of $2(n-3)$ interior triangles between triangular prisms. Therefore, there are $2(n-2) + 2(n-3) = 4n-10$ interior triangles which contribute $4n-10$ to $\sum_{i=1}^{3(n-2)} \alpha_2(T_i)$ but not to $\alpha_2(BQ)$.

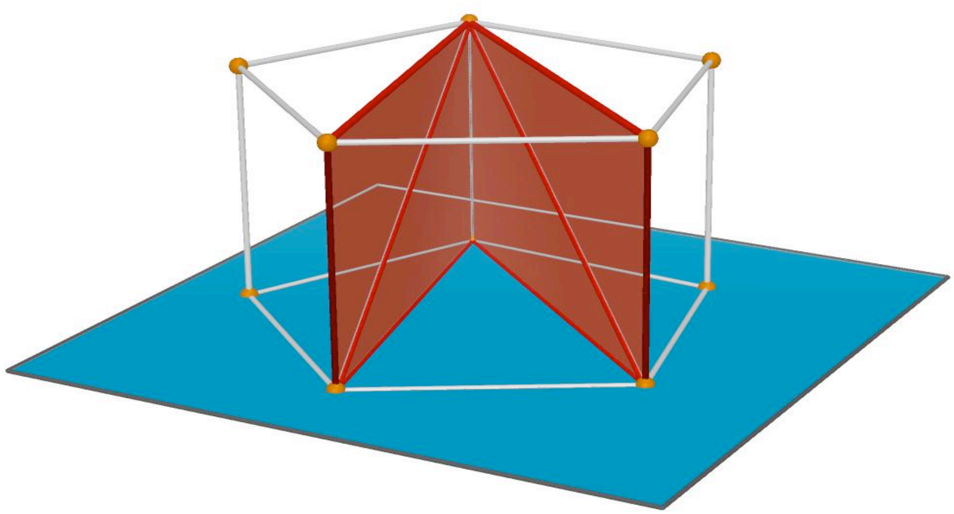

Figure 5.4: 4 faces formed between triangular prisms

Lastly, we must consider the formations of additional exterior faces. When we consider the base triangles formed from subdivision, there are two bases, contributing ½ towards the angle sum at the faces, and $n-3$ additional triangles in each base. Furthermore, when we formed the interior simplices from each triangular prism, every exterior face aside from the two bases was split into two faces each, thus there are n additional faces formed around the exterior. Therefore, there are $2(n-3)+n$ new exterior faces which equals $3n-6$, which contribute $\frac{3n-6}{2}$ to $\sum_{i=1}^{3(n-2)} \alpha_2(T_i)$.

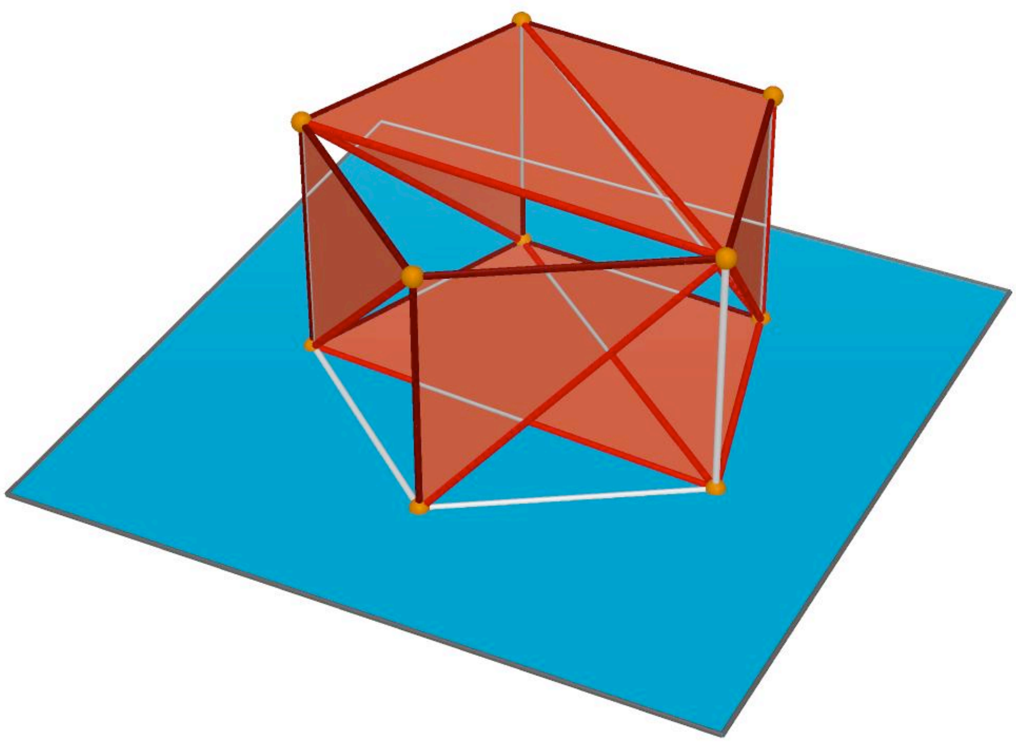

Figure 5.5: Additional exterior faces

So, $4n-10+\frac{3n-6}{2}$, or $\frac{11n-26}{2}$ is subtracted from $\sum_{i=1}^{3(n-2)}\alpha_2(T_i)$, where *4n – 10* represents the interior faces and $\frac{3n-6}{2}$ represents additional exterior faces. Thus, we have

$$\alpha_2(BQ)=\sum_{i=1}^{3(n-2)}\alpha_2(T_i)-\frac{11n-26}{2}.$$

Now, we will consider $\alpha_1(BQ)$. When the polytope is subdivided into *n – 2* triangular prisms, there are *n – 3* additional edges formed on a single base, thus *2(n – 3)* additional edges formed on both bases. The angles of each new edge in a base will sum to ½, since the additional edges we are considering exist on a face. The angle at every face is always ½, so the angle at edges lying on a face must also sum to ½. Thus, the additional angles at newly formed edges on the bases will be $\frac{2(n-3)}{2}=n-3$. Since there are *n* edges formed on the sides of the prism, using the same argument as above, an angle of $\frac{n}{2}$ will be added to $\alpha_1(BQ)$ as well.

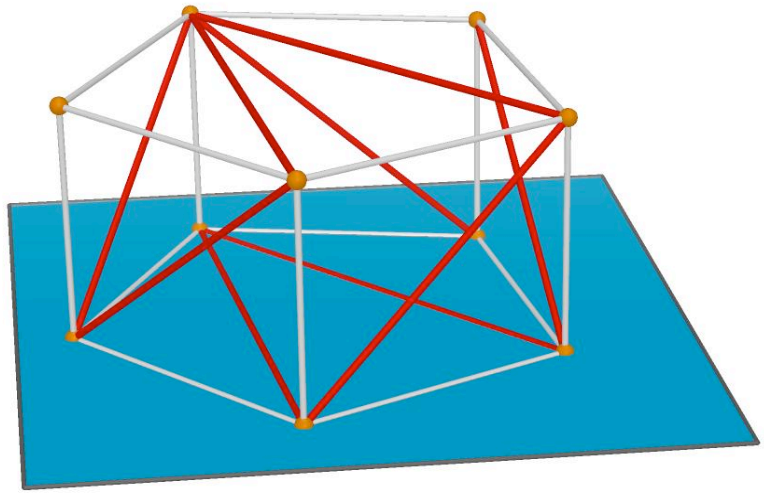

Figure 5.6: Additional exterior edges

In the interior, there is one edge formed on each vertical face which subdivides the polytope into triangular prisms. Since one face borders two separate triangular prisms, the edge on the face will also border two triangular prisms, thus every edge between triangular prisms will contribute *1* to $\alpha_1(BQ)$, for an additional angle sum of $n - 3$.

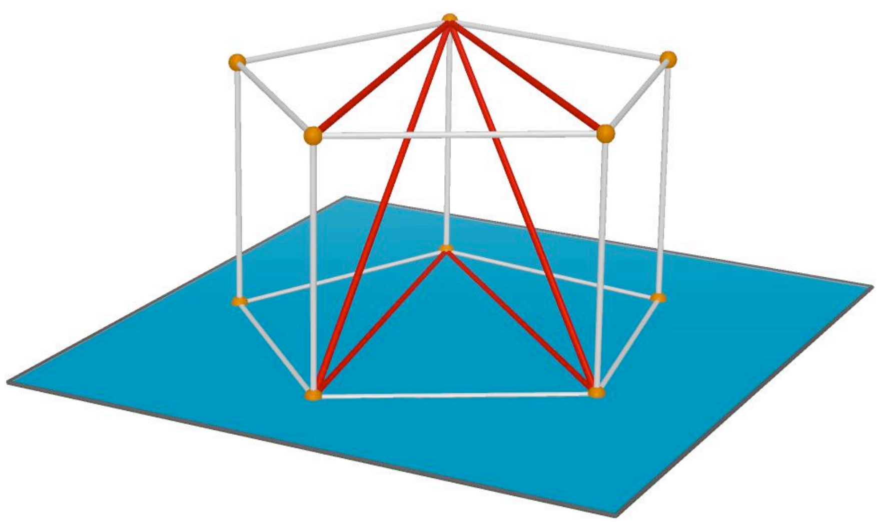

Figure 5.7: Additional interior edges

When we sum $(n-3)+\frac{n}{2}+(n-3)$, we find that $\frac{5n-12}{2}$ has been added to $\alpha_1(BQ)$ in the subdivision. Thus

$$\alpha_1(BQ) = \sum_{i=1}^{3(n-2)} \alpha_1(T_i) - \frac{5n-12}{2}.$$

The argument that $\alpha_0(BQ)$ does not change is the same as the argument that $\alpha_0(PQ)$ does not change. Thus

$$\alpha_0(BQ) = \sum_{i=1}^{3(n-2)} \alpha_0(T_i).$$

Since $\gamma_0(P)$, $\gamma_1(P)$, and $\gamma_3(P)$ are all positive, $\gamma_2(P) = \alpha_1(P) - 2\alpha_0(P)$ is the only case that might affect negativity. First, we will use the fact that $\alpha_1(T) = \alpha_0(T) + 1$ so we can express $\gamma_2(BQ)$ in terms of $\alpha_0(BQ)$. Observe:

$$\alpha_1(BQ) = \sum_{i=1}^{3(n-2)} \alpha_1(T_i) - \frac{5n-12}{2}$$

$$= \sum_{i=1}^{3(n-2)} (\alpha_0(T_i)+1) - \frac{5n-12}{2}$$

$$= \sum_{i=1}^{3(n-2)} \alpha_0(T_i) + 3(n-2) - \frac{5n-12}{2}$$

$$= \sum_{i=1}^{3(n-2)} \alpha_0(T_i) + \frac{n}{2}.$$

We want to show $\gamma_2(BQ) \geq 0$. We can substitute the above into the equation for $\gamma_2(BQ)$ along with $\alpha_0(BQ) = \sum_{i=1}^{3(n-2)} \alpha_0(T_i)$.

$$\begin{aligned}\gamma_2(BQ) &= \alpha_1(BQ) - 2\alpha_0(BQ) \\ &= \sum_{i=1}^{3(n-2)} \alpha_0(T_i) + \frac{n}{2} - 2\sum_{i=1}^{3(n-2)} \alpha_0(T_i) \\ &= \frac{n}{2} - \sum_{i=1}^{3(n-2)} \alpha_0(T_i).\end{aligned}$$

Recall that $\alpha_0(T_i) < ½$ from Chapter 2. This implies that $\sum_{i=1}^{3(n-2)} \alpha_0(T_i) < \frac{3(n-2)}{2}$. We add every α_0 for each simplex on the left, whereas on the right we have precisely $3(n-2)$ simplices times ½.

$$\gamma_2(BQ) = \frac{n}{2} - \sum_{i=1}^{3(n-2)} \alpha_0(T_i) > \frac{n}{2} - \frac{3(n-2)}{2} = -n + 3.$$

This result is inconclusive at the present time. Since this particular method of subdividing an *n* sided prism into multiple triangular prisms yields no conclusive results, some other method may produce results (See Chapter 4).

Proposition 2:

For all 3-dimensional prisms BQ, $\gamma_1(BQ) + \gamma_2(BQ) = \frac{n}{2}$, where n represents the number of sides in the base pyramid.

Proof:

Consider each respective $\gamma_i(BQ)$ in terms of $\alpha_i(BQ)$. We have

$$\gamma_1(BQ) + \gamma_2(BQ) = \alpha_1(BQ) - \alpha_0(BQ).$$

We then have

$$\begin{aligned}\gamma_1(BQ) + \gamma_2(BQ) &= \alpha_1(BQ) - \alpha_0(BQ) \\ &= \sum_{i=1}^{3(n-2)} \alpha_0(T_i) + \frac{n}{2} - \sum_{i=1}^{3(n-2)} \alpha_0(T_i) \\ &= \frac{n}{2}.\end{aligned}$$

Thus, for all 3-dimensional prisms,

$$\gamma_1(BQ) + \gamma_2(BQ) = \frac{n}{2}.$$

□

Chapter 6
Further Conjectures

Our current understanding of the γ-vector is fairly limited. This honors project was an attempt to expand our knowledge of the subject by showing non-negativity of prisms and pyramids for the γ-vector. Our attempts at proving non-negativity of the γ-vector for prisms were inconclusive, yet may be justified through further experimentation. While we have shown non-negativity of the γ-vector for pyramids, we would like to show that all polytopes in *3*-dimensions have a non-negative γ-vector. Furthermore, we would like to show that all *d*-polytopes have a non-negative γ-vector.

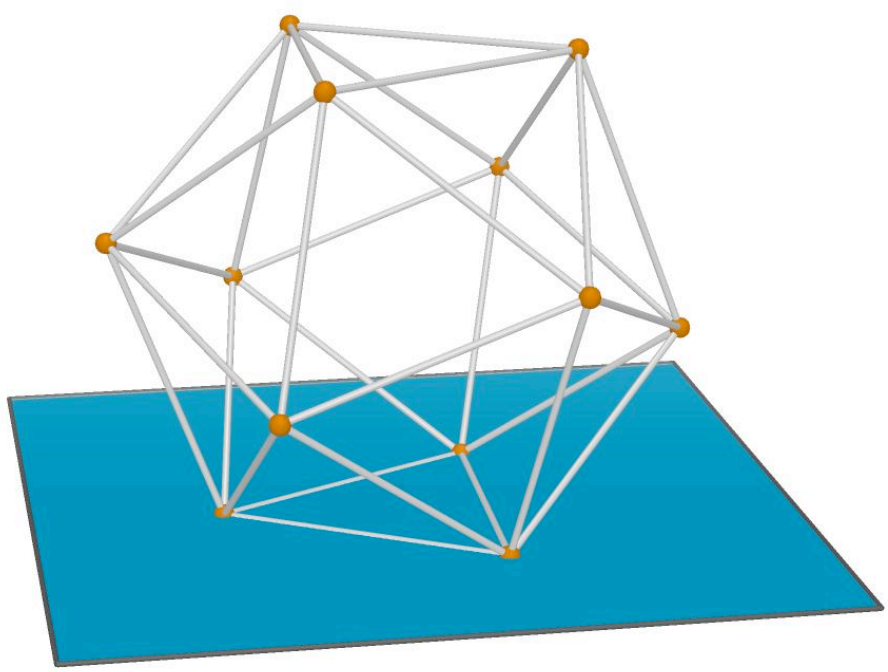

Figure 6.1: 3-dimensional icosahedron is a 3-polytope that is not a prism or pyramid

It is also conjectured that the γ-vector is non-decreasing in *d*-dimensions. In the *3*-dimensional pyramid case we believe that understanding $\gamma_2 > ½$ is a firm starting point to prove the conjecture that the γ-vector is non-decreasing for *3*-dimensional pyramids.

Our ultimate hope is that the γ-vector physically measures something. It is believed that the γ-vector of simplices may measure probabilities, since all data we have reflects $\gamma_i(T)$ values between *0* and *1*. We will continue to experiment and derive conjectures based on available data.

Appendix A

3-Simplex

α_0	α_1	α_2	α_3	α_0 - α_1	γ_0	γ_1	γ_2	γ_3	$\gamma_1 + \gamma_2$
0.19639	1.19639	2	1	1	0	0.19639	0.80361	1	1
0.195026	1.195026	2	1	1	0	0.195026	0.804974	1	1
0.193677	1.193677	2	1	1	0	0.193677	0.806323	1	1
0.192348	1.192348	2	1	1	0	0.192348	0.807652	1	1
0.191042	1.191042	2	1	1	0	0.191042	0.808958	1	1
0.189764	1.189764	2	1	1	0	0.189764	0.810236	1	1
0.188515	1.188515	2	1	1	0	0.188515	0.811485	1	1
0.1873	1.1873	2	1	1	0	0.1873	0.8127	1	1
0.18612	1.18612	2	1	1	0	0.18612	0.81388	1	1
0.184977	1.184977	2	1	1	0	0.184977	0.815023	1	1
0.183874	1.183874	2	1	1	0	0.183874	0.816126	1	1
0.182812	1.182812	2	1	1	0	0.182812	0.817188	1	1
0.181792	1.181792	2	1	1	0	0.181792	0.818208	1	1
0.180815	1.180815	2	1	1	0	0.180815	0.819185	1	1
0.179883	1.179883	2	1	1	0	0.179883	0.820117	1	1
0.178995	1.178995	2	1	1	0	0.178995	0.821005	1	1
0.178152	1.178152	2	1	1	0	0.178152	0.821848	1	1
0.177355	1.177355	2	1	1	0	0.177355	0.822645	1	1
0.176602	1.176602	2	1	1	0	0.176602	0.823398	1	1
0.175895	1.175895	2	1	1	0	0.175895	0.824105	1	1
0.175232	1.175232	2	1	1	0	0.175232	0.824768	1	1
0.174613	1.174613	2	1	1	0	0.174613	0.825387	1	1
0.174037	1.174037	2	1	1	0	0.174037	0.825963	1	1
0.173504	1.173504	2	1	1	0	0.173504	0.826496	1	1
0.173012	1.173012	2	1	1	0	0.173012	0.826988	1	1
0.172561	1.172561	2	1	1	0	0.172561	0.827439	1	1
0.17215	1.17215	2	1	1	0	0.17215	0.82785	1	1
0.171777	1.171777	2	1	1	0	0.171777	0.828223	1	1
0.171442	1.171442	2	1	1	0	0.171442	0.828558	1	1
0.171143	1.171143	2	1	1	0	0.171143	0.828857	1	1
0.170879	1.170879	2	1	1	0	0.170879	0.829121	1	1
0.170649	1.170649	2	1	1	0	0.170649	0.829351	1	1
0.168857	1.168857	2	1	1	0	0.168857	0.831143	1	1
0.164635	1.164635	2	1	1	0	0.164635	0.835365	1	1
0.160535	1.160535	2	1	1	0	0.160535	0.839465	1	1
0.156554	1.156554	2	1	1	0	0.156554	0.843446	1	1
0.152688	1.152688	2	1	1	0	0.152688	0.847312	1	1
0.148936	1.148936	2	1	1	0	0.148936	0.851064	1	1
0.145292	1.145292	2	1	1	0	0.145292	0.854708	1	1
0.141755	1.141755	2	1	1	0	0.141755	0.858245	1	1
0.138321	1.138321	2	1	1	0	0.138321	0.861679	1	1

SOURCES

Bayer, M.M., Billera, L.J.
Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets. Invent. Math., 79(1), 1985, 143-157.

Camenga, K.
Angle Sums on Polytopes and Polytopal Complexes, Ph.D. Dissertation in Mathematics, Cornell University, 2006.

Grunbaum, B.
Convex Polytopes, Graduate Texts in Mathematics 221. Springer-Verlag: New York, 2003.

Perles, M.A., Shephard, G.C.
Angle Sums of Convex Polytopes. Math. Scand., 21(1967), 199-218.

Welzl, E.
Gram's equation-a probabilistic proof, in Results and Trends in Theoretical Computer Science (Graz, 1994), Lecture Notes in Comput. Sci., 812, . Berlin: Springer, 1994, 422-424.

Ziegler, G. M.
Lectures on Polytopes, Graduate Texts in Mathematics. Springer-Verlag: New York, 1995.